Basic Properties of Closure Spaces

Bärbel M. R. Stadler[†], Peter F. Stadler[†,‡,*]

[†]Institut für Theoretische Chemie, Universität Wien, Währingerstraße 17, A-1090 Wien, Austria

[‡]The Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA

[*]Address for corresponce

Abstract. This technical report summarized facts from the basic theory of generalized closure spaces and gives detailed proofs for them. Many of the results collected here are well known for various types of spaces. We have made no attempt to find the original proofs.

1. Set-Valued Set-Functions

1.1. Closure, Interior, Neighborhood, and Convergent. In this section, which in part generalizes the results of Day [8], Hammer [18, 12] and Gniłka [13] on extended topologies, we explore the surprising fact that some meaningful topological concepts can already be defined on a set X endowed with an arbitrary set-valued set-function, which we will interpret as a generalized closure operator.

More formally, let X be a set, $\mathcal{P}(X)$ its power set (i.e., the set of all subsets of X), and let $\mathsf{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ be an arbitrary function. We shall see that it is fruitful to interpret cl as a *closure* function on X; hence we call $\mathsf{cl}(A)$ is the *closure* of the set A. In order to simplify the notation in the following we write $-A$ instead of $X \setminus A$ for the complement of A in X.

The dual of the closure function is the *interior function* $\mathsf{int} : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$\mathsf{int}(A) = -\mathsf{cl}(-A) \tag{1}$$

Given the interior function, we obviously recover the closure as $\mathsf{cl}(A) = -(\mathsf{int}(-A))$. A set $A \in \mathcal{P}(X)$ is *closed* if $A = \mathsf{cl}(A)$ and *open* if $A = \mathsf{int}(A)$. In contrast to "classical" topology, open and closed sets will not play a central role in our discussion. Furthermore, we emphasize that the distinction of closure and interior is completely arbitrary in the absence of additional conditions.

Definition 1. *Let* cl *and* int *be closure function and its dual interior function on X. Then the* neighborhood function $\mathcal{N} : X \to \mathcal{P}(\mathcal{P}(X))$ *and the* convergent function $\mathcal{N}^* : X \to \mathcal{P}(\mathcal{P}(X))$ *assign to each $x \in X$ the collections*

$$\mathcal{N}(x) = \big\{N \in \mathcal{P}(X)\big| x \in \mathsf{int}(N)\big\}$$
$$\mathcal{N}^*(x) = \big\{Q \in \mathcal{P}(X)\big| x \in \mathsf{cl}(Q)\big\} \tag{2}$$

of its neighborhoods *and* convergents, *respectively.*

The convergent function was introduced by G. Gastl and P.C. Hammer [12, 20]. It is not hard to see that neighborhoods and convergents are equivalent:

Theorem 1. $Q \in \mathcal{N}^*(x) \iff (-Q) \notin \mathcal{N}(x)$.

Proof. We have $N \in \mathcal{N}(x)$ if and only if $x \in \mathsf{int}(N) = -\mathsf{cl}(-N)$, i.e., $x \notin \mathsf{cl}(-N)$. On the other hand, $Q \in \mathcal{N}^*(x)$ iff $x \in \mathsf{cl}(Q)$. Thus $x \in \mathsf{cl}(-N)$ if and only if $(-N) \in \mathcal{N}^*(x)$. In other words, $x \notin \mathsf{cl}(-N)$, i.e., $x \in -\mathsf{cl}(-N) = \mathsf{int}(N)$ if and only if $(-N) \notin \mathcal{N}^*(x)$. Using the definition of the neighborhoods we finally have $N \in \mathcal{N}(x)$ if and only if $(-N) \notin \mathcal{N}^*(x)$. $\square$

The next result, which is mentioned for instance in [8], shows that closure and neighborhood are equivalent. Hence given one of closure function cl, interior function int, neighborhood function $\mathcal{N}$ or covergent function $\mathcal{N}^*$, the other three functions are unambiguously defined.

Theorem 2. *Let $\mathcal{N}$ be the neighborhood function defined in equ.(2). Then*

$$x \in \mathsf{cl}(A) \iff (-A) \notin \mathcal{N}(x) \qquad and \qquad x \in \mathsf{int}(A) \iff (-A) \notin \mathcal{N}^*(x) \tag{3}$$

Proof. $x \in \mathsf{cl}(A) = \mathsf{cl}(-(-A))$ if and only if $(-A) \in \{N | x \in \mathsf{cl}(-N)\}$ if and only if $(-A) \notin \{N | x \notin \mathsf{cl}(-N)\} = \{N | x \in (-\mathsf{cl}(-N))\} = \{N | x \in \mathsf{int}(N)\} = \mathcal{N}(x)$.
$x \in \mathsf{int}(A) = \mathsf{int}(-(-A))$ if and only if $(-A) \in \{Q | x \in \mathsf{int}(-N)\}$ if and only if $(-A) \notin \{Q | x \notin \mathsf{int}(-N)\} = \{Q | x \in (-\mathsf{int}(-N))\} = \{Q | x \in \mathsf{cl}(N)\} = \mathcal{N}^*(x)$. $\square$

1.2. Comparison of Closure Spaces. Let c' and c'' be two generalized closure operators on X. We say that c' is *finer* than c'', $c' \succeq c''$, or c'' is *coarser* than c' if $c'(A) \subseteq c''(A)$ for all $A \in \mathcal{P}(X)$. Note that $c' \succeq c''$ and $c' \preceq c''$ implies $c' = c''$.

Theorem 3. *Let c' and c'' be two closure function on X. Denote the associated interior, neighborhood, and convergent functions by i', i'', $\mathcal{N}'$, $\mathcal{N}''$, $\mathcal{N}^{*\prime}$, and $\mathcal{N}^{*\prime\prime}$, respectively. Then the following conditions are equivalent:*

    (i) $c'(A) \subseteq c''(A)$ *for all $A \in \mathcal{P}(X)$.*
    (ii) $i''(A) \subseteq i'(A)$ *for all $A \in \mathcal{P}(X)$.*
    (iii) $\mathcal{N}''(x) \subseteq \mathcal{N}'(x)$ *for all $x \in X$.*
    (iv) $\mathcal{N}^{*\prime}(x) \subseteq \mathcal{N}^{*\prime\prime}(x)$ *for all $x \in X$.*

Proof. We have $c'(-A) \subseteq c''(-A)$ iff $i''(A) = -c''(-A) \subseteq -c'(-A) - i'(A)$ for all A. Using the relationship of closure and neighborhood we may argue $c'(A) \subseteq c''(A)$ iff $x \in c'(A) \implies x \in c''(A)$ iff $-A \notin \mathcal{N}'(x) \implies -A \notin \mathcal{N}''(x)$ iff $-A \in \mathcal{N}''(x) \implies$

$-A \in \mathcal{N}'(x)$ iff $A \in \mathcal{N}''(x) \implies A \in \mathcal{N}'(x)$ for all $A \in \mathcal{P}(X)$. The condition for the converents immediately follows from the duality. $\qquad\square$

If one of the four equivalent conditions in theorem 3 is satisfied we say that (X, c') is *finer* than (X, c'') and that (X, c'') is coarser than (X, c') and we write $(X, c') \succeq (X, c'')$ or $(X, c'') \preceq (X, c')$.

1.3. Pre-Convergence. An alternative approach to defining very weak generalized topological spaces starts with convergence as the basic ingredient. The discussion here generalized the approaches of Choquet, Fisher, Kent and others [6, 24, 10, 22, 26] which is based on Cartan's concept of a *filter* [4, 3] or its generalizations [35], see Appendix A. A different approach to convergence based on Moore-Smith systems [25] is explored in [36].

Definition 2. *A pre-convergence is a relation* q *on* $\mathcal{P}(\mathcal{P}(X)) \times X$ *such that*
(C0) $\mathcal{F} \subseteq \mathcal{G}$ *and* $(\mathcal{F}, x) \in$ q *implies* $(\mathcal{G}, x) \in$ q.

We write usually $\mathcal{F} \to x$ instead of $(\mathcal{F}, x) \in$ q and interpret this symbol as "the collection $\mathcal{F}$ of subsets of X converges to x. The axiom (C0) thus reads "$\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{F} \to x$ implies $\mathcal{G} \to x$".

There is are natural closure and interior operators cq and iq associated with q:
$$\begin{aligned}
\mathsf{iq}(A) &= \{x \in X | \mathcal{F} \to x \implies A \in \mathcal{F}\} \\
\mathsf{cq}(A) &= \{x \in X | \exists F : \mathcal{F} \to x \quad \text{and} \quad (-A) \notin \mathcal{F}\}
\end{aligned} \tag{4}$$
It remains to check that iq and cq are indeed dual: $x \notin \mathsf{cq}(-A)$ iff $\mathcal{F} \to x$ implies $(-(-A)) = A \in \mathcal{F}$. The definition of $\mathsf{iq}(A)$ looks as usual, while the expression for $\mathsf{cq}(A)$ looks somewhat unfamiliar.

We define the q-neighborhood of a point $x \in X$ by
$$\mathcal{N}_{\mathsf{q}}(x) = \bigcup \{\mathcal{F} : \mathcal{F} \to x\} \tag{5}$$
Obviously, $\mathcal{F} \to x$ implies $\mathcal{N}_{\mathsf{q}}(x) \subseteq \mathcal{F}$. Furthermore, if $\mathcal{N}_{\mathsf{q}}(x) \to x$ then $\mathcal{F} \to x$ if and only if $\mathcal{N}_{\mathsf{q}}(x) \subseteq \mathcal{F}$.

Definition 3. *A pre-convergence* q *is ideal if*
(C*) $\mathcal{N}_{\mathsf{q}}(x) \to x$ *for all* $x \in X$.

The definition of the interior reduces to $\mathsf{iq}(A) = \{x | A \in \mathcal{N}_{\mathsf{q}}(x)\}$ in the case of ideal pre-convergences. For the closure we obtain $\mathsf{cq}(A) = \{x | (-A) \notin \mathcal{N}_{\mathsf{q}}(x)\}$, i.e., $\mathsf{cq}(A) = \{x | A \in \mathcal{N}_{\mathsf{q}}^*(x)\}$, where $\mathcal{N}_{\mathsf{q}}^*(x)$ is the dual of the q-neighborhood function. Thus interior, closure, neighborhood and convergent equivalently define an ideal per-convergence relation and *vice versa*.

1.4. Continuity.

Definition 4. *A function* $f : (X, \mathsf{cl}) \to (Y, \mathsf{cl})$ *is*

closure preserving	*if for all* $A \in \mathcal{P}(X)$ *holds* $f(\mathsf{cl}(A)) \subseteq \mathsf{cl}(f(A))$;
continuous	*if for all* $B \in \mathcal{P}(Y)$ *holds* $\mathsf{cl}(f^{-1}(B)) \subseteq f^{-1}(\mathsf{cl}(B))$.

It is obvious that the identity $\imath : (X, \mathsf{cl}) \to (X, \mathsf{cl}) : x \mapsto x$ is both closure-preserving and continuous since $\imath(\mathsf{cl}(A)) = \mathsf{cl}(A) \subseteq \mathsf{cl}(A) = \mathsf{cl}(\imath(A))$. Furthermore, the concatenation $h = g(f)$ of the closure-preserving (continuous) functions $f : X \to Y$ and $g : Y \to Z$ is again closure-preserving (continuous):

$$h(\mathsf{cl}(A)) = g(f(\mathsf{cl}(A))) \subseteq g(\mathsf{cl}(f(A))) \subseteq \mathsf{cl}(g(f(x))) = \mathsf{cl}(h(x))$$

$$\mathsf{cl}\big(h^{-1}(B)\big) = \mathsf{cl}\big(f^{-1}(g^{-1}(B))\big) \subseteq f^{-1}\big(\mathsf{cl}\big(g^{-1}(B)\big)\big) \tag{6}$$

$$\subseteq f^{-1}(g^{-1}(\mathsf{cl}(B))) = h^{-1}(\mathsf{cl}(B))$$

as a consequence of the continuity of f and g.

Theorem 4. *Let (X, cl) and (Y, cl) be two sets with arbitrary closure functions and let $f : X \to Y$. Then the following conditions (for continuity) are equivalent:*

(i) $\mathsf{cl}(f^{-1}(B)) \subseteq f^{-1}(\mathsf{cl}(B))$ *for all $B \in \mathcal{P}(Y)$.*
(ii) $f^{-1}(\mathsf{int}(B)) \subseteq \mathsf{int}(f^{-1}(B))$ *for all $B \in \mathcal{P}(Y)$.*
(iii) $B \in \mathcal{N}(f(x))$ *implies* $f^{-1}(B) \in \mathcal{N}(x)$ *for all $B \in \mathcal{P}(Y)$ and all $x \in X$.*
(iv) $f^{-1}(B) \in \mathcal{N}^*(x)$ *implies* $B \in \mathcal{N}^*(f(x))$ *for all $B \in \mathcal{P}(Y)$ and all $x \in X$.*

Conditions (iii) and (iv) are equivalent for each individual $x \in X$ as well.

Proof. The result is given without the (simple) proof in [15, Thm.3.1.].
We repeatedly use the identity $f^{-1}(U) = -f^{-1}(-U)$ and the equivalence of $A \subseteq A'$ and $-A' \subseteq -A$. We first show that (i) implies (ii) and then the converse:

$$f^{-1}(\mathsf{int}(B)) = -f^{-1}(-\mathsf{int}(B)) = -f^{-1}(\mathsf{cl}(-B)) \subseteq -\mathsf{cl}\big(f^{-1}(-B)\big)$$

$$= -\mathsf{cl}\big(-f^{-1}(B)\big) = \mathsf{int}\big(f^{-1}(B)\big).$$

$$\mathsf{cl}\big(f^{-1}(B)\big) = -(-\mathsf{cl}\big(f^{-1}(B)\big)) = -\mathsf{int}\big(-f^{-1}(B)\big) = -\mathsf{int}\big(f^{-1}(-B)\big)$$

$$\subseteq -f^{-1}(\mathsf{int}(-B)) = -f^{-1}(-\mathsf{cl}(B)) = f^{-1}(\mathsf{cl}(B)).$$

By definition we have

$$\mathsf{int}\big(f^{-1}(B)\big) = \{x \in X | f^{-1}(B) \in \mathcal{N}(x)\}$$

$$f^{-1}(\mathsf{int}(B)) = \{x \in X | B \in \mathcal{N}(f(x))\} \tag{7}$$

$$\mathsf{cl}\big(f^{-1}(B)\big) = \{x \in X | f^{-1}(B) \in \mathcal{N}^*(x)\}$$

$$f^{-1}(\mathsf{cl}(B)) = \{x | B \in \mathcal{N}^*(f(x))\}$$

By (ii) f is continuous if and only if $x \in f^{-1}(\mathsf{int}(B))$ implies $x \in \mathsf{int}(f^{-1}(B))$ for all $B \in \mathcal{P}(Y)$ and all $x \in X$. Using equ.(7) this translates to condition (iii), while (i) translates to (iv). It remains to show that (iii) and (iv) are equivalent for a given $x \in X$:
$f^{-1}(B) \in \mathcal{N}^*(x)$ implies $B \in \mathcal{N}^*(f(x))$ iff $B \notin \mathcal{N}(f(x))$ implies $f^{-1}(B) \notin \mathcal{N}^*(x)$, Equivalently, $(-B) \notin \mathcal{N}(f(x))$ implies $f^{-1}(-B) = -f^{-1}(B) \notin \mathcal{N}^*(x)$ since the conditions must hold for all $B \in \mathcal{P}(Y)$. Now we use the duality of neighborhoods and convergents and obtain the equivalent condition: $B \in \mathcal{N}(f(x))$ implies $f^{-1}(B) \in \mathcal{N}(x)$ for all $B \in \mathcal{P}(Y)$. $\square$

The last part of the theorem gives rise to the

Definition 5. *Let (X, cl) and (Y, cl) be two sets with arbitrary closure functions. Then $f : X \to Y$ is* continuous *in x if for all $B \in \mathcal{P}(Y)$, $B \in \mathcal{N}(f(x))$ implies $f^{-1}(B) \in \mathcal{N}(x)$ (or, $f^{-1}(B) \in \mathcal{N}^*(x)$ implies $B \in \mathcal{N}^*(f(x))$).*

An immediate consequence of theorem 4 is the following familiar relationship between local and global continuity:

Corollary 1. *Let (X, cl) and (Y, cl) be two sets with arbitrary closure functions. Then $f : X \to Y$ is continuous if and only if it is continuous in x for all $x \in X$.*

Convergence gives rise to its own "natural" version of continuity. Consider two pre-convergence spaces (X, x) and (X, y) and $f : X \to Y$. The most natural definition of q-continuity would be to require that $f(\mathcal{F}) \to_{\mathsf{y}} f(x)$ implies $f(\mathcal{F}) \to_{\mathsf{y}} f(x)$. This definition behaves reasonably only when $f(X) = Y$. Otherwise, no function could be continuous whenever there is a convergent pre-filter $\mathcal{G}$ on Y that does not satisfy $\mathcal{G} \subseteq \mathcal{P}(f(X))$, i.e., continuity would be destroyed by the large neighborhoods in Y.

Theorem 5. *(X, c') is finer than (X, c''), $(X, c') \succeq (X, c'')$, if and only if $\imath : (X, c') \to (X, c'')$ is continuous.*

Proof. By definition $\imath$ is continuous if and only if $c'(\imath^{-1}(B)) \subseteq \imath^{-1}(c''B)$ for all $B \in \mathcal{P}(X)$. Since $\imath(x) = \imath^{-1}(x) = x$ this condition simplifies to $c'(B) \subseteq c''(B)$, i.e., to the definition of $(X, c') \succeq (X, c'')$. $\qquad\square$

The definition of finer and coarser structures in section 1.2 thus coincides with the category theoretic notion, see e.g. [28].

1.5. Neighborhoods of Sets. The notation of a neighborhood for an individual point can be extended naturally to sets.

Definition 6. *Let $A \in \mathcal{P}(X)$. A set V is a* neighborhood *of A, in symbols $V \in \mathcal{N}(A)$ if $V \in \mathcal{N}(x)$ for all $x \in A$.*

Obviously $\mathcal{N}(\{x\}) = \mathcal{N}(x)$.

Lemma 1. *For all $V, A \in \mathcal{P}(X)$ we have $V \in \mathcal{N}(A)$ iff and only if $A \subseteq \mathsf{int}(()\, V)$.*

Proof. $V \in \mathcal{N}(A)$ iff $\forall x \in A : V \in \mathcal{N}(x)$ iff $\forall x \in A : x \mathsf{in} \mathsf{int}(V)$, i.e., iff $x \in A$ implies $x \in \mathsf{int}(V)$. $\qquad\square$

1.6. Limit Points.

Definition 7. *A point p is a* limit point *of $A \subseteq X$ if each neighborhood $N \in \mathcal{N}(p)$ satisfies $N \cap (A \setminus \{p\}) \neq \emptyset$. The set*

$$A^{\blacktriangledown} = \{x \in X \,|\, \forall\, N \in \mathcal{N}(x) : N \cap (A \setminus \{p\}) \neq \emptyset\} \qquad (8)$$

of all limit points of A is called the derived set *of A.*

Derived sets are at the basis of Sierpiński's presentation of generalized topologies [31].

Note that if x has no neighborhood, $\mathcal{N}(x) = \varnothing$, then $x \in \mathsf{cl}(A)$ and $x \in A^{\blacktriangledown}$ for all sets $A \subseteq X$. Thus $\emptyset^{\blacktriangledown} = \{x \in X | \mathcal{N}(x) = \varnothing\}$.

The definition immediately implies

$$x \in A^{\blacktriangledown} \quad \implies \quad x \in (A \setminus \{x\})^{\blacktriangledown} \tag{9}$$

If $B \subseteq A$ we have $B^{\blacktriangledown} \subseteq A^{\blacktriangledown}$ since $N \cap (B \setminus \{p\}) \neq \emptyset$ obviously implies $N \cap (A \setminus \{p\}) \neq \emptyset$. This property is called isotony and will be the focus of the following section.

Definition 8. *A set $A \subseteq X$ is* dense-in-itself *if $A \subseteq A^{\blacktriangledown}$.*

2. Isotonic Spaces

2.1. Isotony and Stacks. Almost all approaches to extend the framework of topology at least assume that the closure functions are isotonic, or, equivalently, that the neighborhoods of a point form a "stack", [17, 18, 8, 2, 13] and many others. The importance of isotony is emphasized by a large number of equivalent conditions.

Lemma 2. [18, Lem.10] *The following conditions are equivalent for arbitrary functions $\mathsf{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$.*

(K1) $A \subseteq B$ *implies* $\mathsf{cl}(A) \subseteq \mathsf{cl}(B)$ *for all* $A, B \in \mathcal{P}(X)$.
(K1') $\mathsf{cl}(A) \cup \mathsf{cl}(A) \subseteq \mathsf{cl}(A \cup B)$ *for all* $A, B \in \mathcal{P}(X)$.
(K1") $\mathsf{cl}(A \cap B) \subseteq \mathsf{cl}(A) \cap \mathsf{cl}(B)$

A closure function satisfying (K1) is called *isotonic.*

Proof. Suppose $A \subseteq B$ implies $\mathsf{cl}(A) \subseteq \mathsf{cl}(B)$. Then $A, B \subseteq A \cup B$ for all A and B implies $\mathsf{cl}(A) \subseteq \mathsf{cl}(A \cup B)$, $\mathsf{cl}(B) \subseteq \mathsf{cl}(A \cup B)$ and therefore $\mathsf{cl}(A) \cup \mathsf{cl}(B) \subseteq \mathsf{cl}(A \cup B)$. Analogously, $A \cap B \subseteq A, B$ implies $\mathsf{cl}(A \cap B) \subseteq \mathsf{cl}(A) \cap \mathsf{cl}(B)$. Next assume $\mathsf{cl}(A) \cup \mathsf{cl}(B) \subseteq \mathsf{cl}(A \cup B)$ and consider $A \subseteq B$. Then $\mathsf{cl}(A) \subseteq \mathsf{cl}(A) \cup \mathsf{cl}(B) \subseteq \mathsf{cl}(A \cup B) = \mathsf{cl}(B)$. Finally, if $\mathsf{cl}(A \cap B) \subseteq \mathsf{cl}(A) \cap \mathsf{cl}(B)$ and $A \subseteq B$ we have $\mathsf{cl}(A) = \mathsf{cl}(A \cap B) = \subseteq \mathsf{cl}(A) \cap \mathsf{cl}(B) \subseteq \mathsf{cl}(B)$. □

It is easy to derive equivalent conditions for the associated interior function by repeated applications of $\mathsf{int}(A) = -\mathsf{cl}(-A)$ and $\mathsf{cl}(A) = -\mathsf{int}(-A)$. One obtains

(K1''') $A \subseteq B$ implies $\mathsf{int}(A) \subseteq B$ for all $A, B \in \mathcal{P}(X)$.
(K1$^{\text{IV}}$) $\mathsf{int}(A) \cup \mathsf{int}(B) \subseteq \mathsf{int}(A \cup B)$ for all $A, B \in \mathcal{P}(X)$.
(K1$^{\text{V}}$) $\mathsf{int}(A \cap B) \subseteq \mathsf{int}(A) \cap \mathsf{int}(B)$ for all $A, B \in \mathcal{P}(X)$.

A (not necessarily non-empty) collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *stack* if $F \in \mathcal{F}$ and $F \subseteq G$ implies $G \in \mathcal{F}$. Let us write $\mathfrak{S}(X)$ for the set of all stacks. It is important to distinguish the empty set $\emptyset \in \mathcal{P}(X)$ and the empty stack $\varnothing \subseteq \mathcal{P}(X)$.

Most interestingly, there is are conditions equivalent to (K1) in terms of neighborhood and convergent functions:

Lemma 3. *The closure functions cl is isotonic if and only if*

(K1ᵛⁱ) $\mathcal{N}(x)$ *is a stack for all $x \in X$.*
(K1ᵛⁱⁱ) $\mathcal{N}^*(x)$ *is a stack for all $x \in X$.*

Proof. We have $N \in \mathcal{N}(x)$ iff $x \in \mathsf{int}(N)$. Thus, if $\mathcal{N}(x)$ is isotonic we have $N' \in \mathcal{N}(x)$ and hence $x \in \mathsf{int}(N')$ for all supersets N' of N. It follows that $N \subset N'$ implies $\mathsf{int}(N) \subseteq \mathsf{int}(N')$. To see the converse, suppose $\mathcal{N}(x)$ is isotonic, $A \subseteq B$, and there is $x \in \mathsf{cl}(A)$ such that $x \notin \mathsf{cl}(B)$. Equivalently, we have $-A \notin \mathcal{N}(x)$, $-B \in \mathcal{N}(x)$, and $-B \subseteq -A$. Isotony of $\mathcal{N}(x)$ thus implies $-A \in \mathcal{N}(x)$, a contradiction. Thus $A \subseteq B$ implies $\mathsf{cl}(A) \subseteq \mathsf{cl}(B)$. Finally, it is well known that $\mathcal{N}(x)$ is a stack if and only if its dual $\mathcal{N}^*(x)$ is stack. □

Isotony (K1) is necessary and sufficient e.g. to replace equ.(3) by a more familiar expression for the closure in terms of neighborhoods: in (pre)topological spaces the closure $c(A)$ is defined by the well-known formula

$$c(A) = \{x \in X | \forall N \in \mathcal{N}(x) : A \cap N \neq \emptyset\} \tag{10}$$

which is meaningful in arbitrary closure spaces. We have the

Theorem 6. [8, Thm.3.1,Cor.3.2] *Let (X, cl) an arbitary closure space.*

  (i) $c(A) \subseteq \mathsf{cl}(A)$ *for all $A \in \mathcal{P}(X)$.*
  (ii) $c : \mathcal{P}(X) \to \mathcal{P}(X)$ *is isotonic.*
  (iii) $c(A) = \mathsf{cl}(A)$ *if and only if $\mathsf{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ is isotonic.*

Proof. We have $N \in \mathcal{N}(x)$ iff $x \in \mathsf{int}(N)$ iff $x \in -\mathsf{cl}(-N)$. Thus $x \in c(A)$, i.e., "$N \in \mathcal{N}(x)$ implies $N \cap A \neq \emptyset$" is equivalent to "$N \cap A = \emptyset$ implies $x \in \mathsf{cl}(-A)$". Now replace $B = -N$ and observe that $(-B) \cap A = \emptyset$ is equivalent with $A \subseteq B$. Thus $x \in c(A)$ iff and only if for all supersets B of A we have $x \in \mathsf{cl}(B)$, i.e.,

$$c(A) = \bigcup_{B:A \subseteq B} \mathsf{cl}(B) \tag{11}$$

$c(A) \subseteq \mathsf{cl}(A)$ and isotony of c follows immediately. We have $c(A) = \mathsf{cl}(A)$ if and only if $\mathsf{cl}(A) \subseteq CB$ for all $A \subseteq B$, i.e., if and only if the closure function cl is isotonic. □

In an isotonic space we also have a more familiar relationship between convergents and neighborhoods. Define

$$\sec \mathcal{F} = \{G \in \mathcal{P}(X) | \forall F \in \mathcal{F} : G \cap F \neq \emptyset\} \tag{12}$$

Theorem 7. [12, Thm.2] *Let (X, cl) be an isotonic space. Then $\mathcal{N}^*(x) = \sec \mathcal{N}(x)$ and $\mathcal{N}(x) = \sec \mathcal{N}^*(x)$.*

Lemma 4. $A^{\blacktriangledown} \subseteq \mathsf{cl}(A)$ *and $A^{\blacktriangledown} - A = \mathsf{cl}(A) - A$.*

Proof. Suppose $p \notin A$. Then $p \in A^{\blacktriangledown}$ iff $\forall N \in \mathcal{N}(p) : N \cap (A - \{p\}) = N \cap A \neq \emptyset$, i.e. iff $p \in \mathsf{cl}(A)$ by theorem 6. This proves the second statement. In general $N \cap (A - \{p\}) \neq \emptyset$ implies $N \cap A \neq \emptyset$, hence $p \in A^{\blacktriangledown}$ implies $p \in \mathsf{cl}(A)$. □

2.2. Stack-Convergence. A much more useful notion of pre-convergence arises by restricting the convergence relation q to $\mathfrak{S}(X) \times X$. Much of the following is discussed in [37] and in particular in [16].

Definition 9. (X, q) *is a* stack-convergence space *if* $q \subseteq \mathfrak{S}(X) \times X$ *satisfies*
(C0) $\mathcal{F} \subseteq \mathcal{G}$ *and* $(\mathcal{F}, x) \in q$ *implies* $(\mathcal{G}, x) \in q$.
(X, q) *is an* ideal stack-convergence space *if in addition*
(C*) $\mathcal{N}_q(x) \to x$ *for all* $x \in X$.
is satisfied.

Obviously, if (X, q) is an ideal stack convergence space then the q-neighborhoods $\mathcal{N}_q(x)$ form a stack. Conversely, if (X, cl) is an isotonic space, we can define a stack-convergence relation qc by setting $(\mathcal{F}, x) \in qc$ iff $\mathcal{N}(x) \subseteq \mathcal{F}$. Thus we have $(\mathcal{F}, x) \in q$ if and only if $\mathcal{N}_q(x) \subseteq \mathcal{F}$ if and only if $(\mathcal{F}, x) \in qc$ if and only if $\mathcal{N}_{qc}(x) \subseteq \mathcal{F}$. Thus we can identify the ideal stack-convergence spaces with the isotonic spaces.

Let (X, x) and (Y, y) be *stack-convergence* spaces and $f : X \to Y$. We define the associated map $f : \mathfrak{S}(X) \to \mathfrak{S}(Y)$ (for which we will always use the same symbol), by
$$f(\mathcal{F}) = \{G \in \mathcal{P}(Y) | \exists F \in \mathcal{F} : f(F) \subseteq G\} \tag{13}$$
It is obvious that $f(\mathcal{F})$ is indeed a stack on Y. Furthermore, we have $f(\mathcal{P}(X)) = \mathcal{P}(Y)$ and $f(\varnothing) = \varnothing$.

2.3. Continuity. Now we are in the position to define stack-continuity in a familiar form:

Definition 10. *Let* (X, x) *and* (X, y) *be two stack-convergence spaces. Then* $f : X \to Y$ *is* stack-continuous in x *if*
$$\mathcal{F} \to_x x \quad implies \quad f(\mathcal{F}) \to_y f(x) \tag{14}$$
$f : X \to Y$ *is* stack-continuous *if it is stack-continuous in* x *for each* $x \in X$.

Theorem 8. *Let* (X, x) *and* (X, y) *be ideal stack-convergence spaces. Then* $f : X \to Y$ *is* q-*continuous in* x *if and only if*
$$\mathcal{N}_y(f(x)) \subseteq f(\mathcal{N}_x(x)) \tag{15}$$

Proof. Using theorem 3 we can rewrite definition 10 in the form $\mathcal{N}_x(x) \subseteq \mathcal{F} \implies \mathcal{N}_y(f(x)) \subseteq f(\mathcal{F})$. Substituting $\mathcal{N}_x(x)$ for $\mathcal{F}$ and using the fact that $\mathcal{G} \subseteq \mathcal{F}$ implies $f(\mathcal{F}) \subseteq f(\mathcal{F})$ we get $\mathcal{N}_y(f(x)) \subseteq f(\mathcal{N}_x(x)) \subseteq f(\mathcal{F})$ whenever $\mathcal{N}_x(x) \subseteq \mathcal{F}$. Thus condition (15) is necessary and sufficient. $\square$

The most important observation in this section is that in isotonic spaces the different types of structure-preserving maps are the same. We first show that local continuity and local stack-continuity are equivalent.

Theorem 9. *Let* (X, cl) *and* (Y, cl) *be isotonic spaces and* $f : X \to Y$ *a function. Then the following properties are equivalent:*

(i) f *is continuous in* x.
(ii) f *is stack-continuous in* x.

Proof. Suppose $B \in \mathcal{N}(f(x))$ implies $f^{-1}(B) \in \mathcal{N}(x)$. Then there is $A \in \mathcal{N}(x)$ such that $A \subseteq f^{-1}(B)$. Thus $f(A) \subseteq f(f^{-1}(B)) \subseteq B$ and hence, by isotony, $B \in f(\mathcal{N}(x))$, i.e., $\mathcal{N}(f(x)) \subseteq f(\mathcal{N}(x))$.

Conversely, suppose f is stack-continuous in x. Then $B \in \mathcal{N}(f(x))$ implies that there is $A \in \mathcal{N}(x)$ such that $f(A) \subseteq B$. We have $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(B)$ and hence $f^{-1}(B) \in \mathcal{N}(x)$. Thus f is continuous in x. $\qquad\square$

Results similar to this and the following theorem can be found e.g. in [8, Thm.6.1]. Let us now turn to the global versions of continuity.

Theorem 10. *Let (X, cl) and (Y, cl) be isotonic spaces. Then the following properties are equivalent:*

    (i) *$f : X \to Y$ is continuous.*
    (ii) *$f : X \to Y$ is closure preserving.*
    (iii) *$f(A) \subseteq B$ implies $f(\mathsf{cl}(A)) \subseteq \mathsf{cl}(B)$ for all $A \in \mathcal{P}(X)$ and all $B \in \mathcal{P}(Y)$.*
    (iv) *$f : X \to Y$ is stack-continuous.*

In general, (iii) implies that f is continuous and closure preserving.

Proof. We first show that (iii) implies (i) and (ii) without assuming that cl is isotonic. Set $A = f^{-1}(B)$ then $f(f^{-1}(B)) \subseteq B$, whence (iii) implies $f(\mathsf{cl}(f^{-1}(B))) \subseteq \mathsf{cl}(B)$. We have $\mathsf{cl}(f^{-1}(B)) \subseteq f^{-1}(f(\mathsf{cl}(f^{-1}(B)))) \subseteq f^{-1}(\mathsf{cl}(B))$, i.e., f is continuous. Now assume (iii) and set $B = f(A)$. Since $f(A) \subseteq f(A)$, we have conclude that $f(\mathsf{cl}(A)) \subseteq \mathsf{cl}(f(A))$, i.e., f is closure preserving.

Suppose f is continuous and cl is isotonic. Then

$$f(A) \subseteq B \implies A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(B) \implies \mathsf{cl}(A) \subseteq \mathsf{cl}(f^{-1}(B)) \subseteq f^{-1}(\mathsf{cl}(B))$$
$$\implies f(\mathsf{cl}(A)) \subseteq f(f^{-1}(\mathsf{cl}(B))) \subseteq \mathsf{cl}(B) \implies \text{(iii)}$$

If f is closure preserving and cl is isotonic we argue: $f(A) \subseteq B$ implies $f(\mathsf{cl}(A)) \subseteq \mathsf{cl}(f(A)) \subseteq \mathsf{cl}(B)$, i.e., (iii) is satisfied.
(iv) follows directly from theorem 9. $\qquad\square$

3. Axioms Systems for Closure Spaces

Let (X, cl) be a generalized closure space and consider the following properties of the closure function for all $A, B \in \mathcal{P}(X)$.

    (K0) $\mathsf{cl}(\emptyset) = \emptyset$.
    (K1) $A \subseteq B$ implies $\mathsf{cl}(A) \subseteq \mathsf{cl}(B)$ (isotonic).
    (K2) $A \subseteq \mathsf{cl}(A)$ (expanding).
    (K3) $\mathsf{cl}(A \cup B) \subseteq \mathsf{cl}(A) \cup \mathsf{cl}(B)$ (sub-additive).
    (K4) $\mathsf{cl}(\mathsf{cl}(A)) = \mathsf{cl}(A)$ (idempotent).
    (K5) $\bigcap_{i \in I} \mathsf{cl}(A_i) = \mathsf{cl}\left(\bigcap_{i \in I} A_i\right)$ (additive).

Theorem 11. *The conditions in each row of table 1 are equivalent.*

Further equivalent conditions in terms of convergence in terms of convergence can be found in [16].

Table 1. The basic axioms.
The properties below are meant to hold for all $A, B \in \mathcal{P}(X)$ and all $x \in X$, respectively.

	closure	interior	neighborhood	convergent
K0'	$\exists A : x \notin \mathsf{cl}(A)$	$\exists A : x \in \mathsf{int}(A)$	$\mathcal{N}(x) \neq \varnothing$	$\mathcal{N}^*(x) \neq \mathcal{P}(X)$
K0	$\mathsf{cl}(\emptyset) = \emptyset$	$\mathsf{int}(X) = X$	$X \in \mathcal{N}(x)$	$\emptyset \notin \mathcal{N}^*(x)$
K1 isotonic, monotone	$A \subseteq B \implies \mathsf{cl}(A) \subseteq \mathsf{cl}(B)$ $\mathsf{cl}(A \cap B) \subseteq \mathsf{cl}(A) \cap \mathsf{cl}(B)$ $\mathsf{cl}(A) \cup \mathsf{cl}(B) \subseteq \mathsf{cl}(A \cup B)$	$A \subseteq B \implies \mathsf{int}(A) \subseteq \mathsf{int}(B)$ $\mathsf{int}(A) \cup \mathsf{int}(B) \subseteq \mathsf{int}(A \cup B)$ $\mathsf{int}(A \cap B) \subseteq \mathsf{int}(A) \cap \mathsf{int}(B)$	$N \in \mathcal{N}(x)$ and $N \subseteq N'$ $\implies$ $N' \in \mathcal{N}(x)$	$Q \in \mathcal{N}^*(x)$ and $Q \subseteq Q'$ $\implies$ $Q' \in \mathcal{N}^*(x)$
KA	$\mathsf{cl}(X) = X$	$\mathsf{int}(\emptyset) = \emptyset$	$\emptyset \notin \mathcal{N}(x)$	$X \in \mathcal{N}^*(x)$
KB	$A \cup B = X \implies$ $\mathsf{cl}(A) \cup \mathsf{cl}(B) = X$	$A \cap B = \emptyset \implies$ $\mathsf{int}(A) \cap \mathsf{int}(B) = \emptyset$	$N', N'' \in \mathcal{N}(x) \implies$ $N' \cap N'' \neq \emptyset$	$Q' \cup Q'' = X \implies$ $Q' \in \mathcal{N}^*(x) \vee Q'' \in \mathcal{N}^*(x)$
K2 expansive	$A \subseteq \mathsf{cl}(A)$	$\mathsf{int}(A) \subseteq \mathsf{int}(A)$	$N \in \mathcal{N}(x) \implies x \in N$	$x \in Q \implies Q \in \mathcal{N}^*(x)$
K3 sub-linear	$\mathsf{cl}(A \cup B) \subseteq \mathsf{cl}(A) \cup \mathsf{cl}(B)$	$\mathsf{int}(A) \cap \mathsf{int}(B) \subseteq \mathsf{int}(A \cup B)$	$N', N'' \in \mathcal{N}(x) \implies$ $N' \cap N'' \in \mathcal{N}(x)$	$(Q' \cup Q'') \in \mathcal{N}^*(x) \implies$ $Q' \in \mathcal{N}^*(x) \vee Q'' \in \mathcal{N}^*(x)$
K4 idempotent	$\mathsf{cl}(\mathsf{cl}(A)) = \mathsf{cl}(A)$	$\mathsf{int}(\mathsf{int}(A)) = \mathsf{int}(A)$	$N \in \mathcal{N}(x) \iff$ $\mathsf{int}(N) \in \mathcal{N}(x)$	$Q \in \mathcal{N}^*(x) \iff$ $\mathsf{cl}(Q) \in \mathcal{N}^(x)$
K5 additive	$\bigcup_{i \in I} \mathsf{cl}(A_i) = \mathsf{cl}\left(\bigcup_{i \in I} A_i\right)$	$\bigcap_{i \in I} \mathsf{int}(A_i) = \mathsf{int}\left(\bigcap_{i \in I} A_i\right)$	$\mathcal{N}(x) = \emptyset$ or $\exists N(x):$ $N \in \mathcal{N}(x)$ $\iff N(x) \subseteq N$	?

Proof. Many of the equivalences in table 1 have been noted previously, see e.g. [12]. We collect the proofs here for completeness. The equivalence of the conditions for neighborhoods and convergents follow immediately from discussion in Appendix A.

K0'. $\mathcal{N}(x) \neq emptyset$ iff there is a set A such that $(-A) \in \mathcal{N}(x)$, i.e., $x \notin \mathsf{cl}(A)$, i.e., $x \in -\mathsf{cl}(A) = \mathsf{int}(-A)$.

K0. $X \in \mathcal{N}(x)$ iff $x \in \mathsf{int}(X) = -\mathsf{cl}(-X) = -\mathsf{cl}(\emptyset)$ iff $x \notin \mathsf{cl}(\emptyset)$ for all $x \in X$.

K1. The equivalent definitions of isotony are discussed in section 2.

KA. $X = C(X) = -\mathsf{int}(-X) = -\mathsf{int}(\emptyset)$ iff $\mathsf{int}(\emptyset) = \emptyset$. We have $\emptyset \in \mathcal{N}(x)$ iff $x \in \mathsf{int}(\emptyset)$, i.e., $\emptyset \notin \mathcal{N}(x)$ for all x iff $\mathsf{int}(\emptyset) = \emptyset$.

KB. First note that $A \cup B = X$ iff $(-A) \cap (-B) = \emptyset$. Hence $A \cap B = \emptyset$ implies $-\mathsf{cl}(-A) \cap -\mathsf{cl}(-B) = \emptyset$, i.e., $\mathsf{int}(A) \cap \mathsf{int}(B) = \emptyset$. Recall $x \in \mathsf{cl}(A) \iff A \in \mathcal{N}^*(x)$. Suppose $A \cup B = X$. Then $\mathsf{cl}(A) \cup \mathsf{cl}(B) = X$ means that for each $x \in X$ we have $x \in \mathsf{cl}(A)$ or $x \in \mathsf{cl}(B)$, and equivalently, $A \in \mathcal{N}^*(x)$ or $B \in \mathcal{N}^*(x)$. The condition for the neighborhood follows by duality, see Appendix A.

K2. $A \subseteq \mathsf{cl}(A) \iff -A \subseteq \mathsf{cl}(-A) \iff \mathsf{int}(A) = -\mathsf{cl}(-A) \subseteq -(-A) = A$. Hence $N \in \mathcal{N}(x)$ implies $x \in \mathsf{int}(N) \subseteq N$, i.e., $x \in N$. Conversely, suppose $x \in N$ for all $N \in \mathcal{N}(x)$ and consider $x \in A$. Then $x \notin -A$ and hence $-A \notin \mathcal{N}(x)$, i.e., $x \in \mathsf{cl}(A)$ by equ.(3).

K3. The axiom can be expressed as $x \in \mathsf{cl}(A \cup B)$ implies $x \in \mathsf{cl}(A)$ or $x \in \mathsf{cl}(B)$. Equivalently, $x \notin \mathsf{cl}(A)$ and $x \notin \mathsf{cl}(B)$ implies $x \notin \mathsf{cl}(A \cup B)$. Using equ.(3) three times we see that this is equivalent with: $-A \in \mathcal{N}(x)$ and $-B \in \mathcal{N}(x)$ implies $-(A \cup B) = (-A) \cap (-B) \in \mathcal{N}(x)$. Replacing A and B by $-A$ and $-B$, respectively, completes the proof.

K4. We have $\mathsf{int}(A) = \mathsf{int}(-\mathsf{cl}(-A)) = -\mathsf{cl}(-[-\mathsf{cl}(-A)]) = -\mathsf{cl}(\mathsf{cl}(-A))$. Thus $\mathsf{cl}(\mathsf{cl}(-A)) = \mathsf{cl}(-A)$ implies $\mathsf{int}(\mathsf{int}(A)) = -\mathsf{cl}(-A) = \mathsf{int}(A)$.
Conversely, $\mathsf{int}(\mathsf{int}(A)) = \mathsf{int}(A)$ implies $-\mathsf{cl}(\mathsf{cl}(-A)) = -\mathsf{cl}(-A)$ and $\mathsf{cl}(\mathsf{cl}(A)) = \mathsf{cl}(A)$. We have in general $x \in \mathsf{int}(\mathsf{int}(A)) = -(\mathsf{cl}(\mathsf{cl}(-A)))$ iff $x \notin \mathsf{cl}(\mathsf{cl}(-A))$, i.e., iff $-(\mathsf{cl}(-A)) = \mathsf{int}(A) \in \mathcal{N}(x)$. Thus K4 is equivalent to the condition that $x \in \mathsf{int}(A)$ if and only if $\mathsf{int}(A) \in \mathcal{N}(x)$. On the other hand, we have $x \in \mathsf{int}(A)$ if and only if $A \in \mathcal{N}(x)$. Combining these two conditions we obtain $A \in \mathcal{N}(x)$ if and only if $\mathsf{int}(A) \in \mathcal{N}(x)$.

K5. Since (K1) from (K5) we know from theorem 6 that $x \in \mathsf{cl}(A)$ iff $N \cap A \neq \emptyset$ for all $N \in \mathcal{N}(x)$. In particular, therefore, $x \in \mathsf{cl}(\{y\}) \iff y \in N$ for all $N \in \mathcal{N}(x)$, i.e., if and only if $y \in N(x)$.
Now we have $\mathsf{cl}(A) = \bigcup_{y \in A} \mathsf{cl}(\{y\})$, hence $\mathsf{int}(A) = -(\mathsf{cl}(-A)) = -\bigcup_{y \in -A} \mathsf{cl}(\{y\}) = \bigcap_{y \in -A} -\mathsf{cl}(\{y\})$. Thus $x \in \mathsf{int}(N)$ iff $x \notin \mathsf{cl}(\{y\})$ for all $y \notin N$, i.e., iff $y \notin N(x)$ for all $y \notin N$ i.e., iff $N(x) \subseteq N$. Collecting the equivalences we have $N \in \mathcal{N}(x)$ iff $N(x) \subseteq N$. $\square$

A few remarks are in order here.

(K0) implies (K0').

Table 2. Axioms for closure operators.
Defining axioms are marked by •, further properties that implied are marked by ○.

Axiom	$cl(\emptyset) = \emptyset$	$A \subseteq B \implies cl(A) \subseteq cl(B)$ isotonic	$A \subseteq cl(A)$ enlarging	$cl(A \cup B) \subseteq cl(A) \cup cl(B)$ sub-linear	$cl(cl(A)) = cl(A)$ idempotent	$cl(\bigcup_i A_i) = \bigcup_i cl(A_i)$ additive	Ref.
Extended Topology	•	•					
Brissaud	•		•				[2]
Neighborhood space	•	•	•				[19]
Closure space	(•)	•	•		•		[33]
Smyth space	•	•		•			[32]
Pretopology	•	•	•	•			[5]
Topology	•	○	•	•	•		
Alexandroff space	•	○	•	○		•	
Alexandroff topology	•	○	•	○	•	•	

Isotonic closure functions satisfying (K0') also satisfy (K0). Such closures are studied in detail in [13, 14] under the name *extended topology*.

(KB) and (K0) implies (KA).

(K2) implies (KB).

(K3) implies (KB).

An isotonic sub-linear closure function, (K1) and (K3), is called *linear* or *finitely additive* and satisfied $cl(A \cup B) = CA \cup cl(B)$ and $int(A \cap B) = int(A) \cap IB$. Often, this condition replaces (K1) and (K3) in axiom systems of closure spaces, see e.g. [11].

Axiom (K4) can be rephrased in the more familiar form *Each neighborhood* $N \in \mathcal{N}(x)$ *contains an* open *neighborhood*, namely $int(N) = int(int(N))$. Idempotence of the closure operator, hence, is the distinguishing feature of topological spaces.

By definition, (K5) implies linearity and therefore both (K1) and (K3).

Spaces with the same combination of axioms have different names in the literature and the same name is used for different axiom systems by different authors. Table 2 summarizes the names used in this manuscript.

Dikranjan *et al.* [9] show that the class of generalized closure space (X, cl) satisfying (K0), (K1), (K2), and (K4) form a topological category. It is well known that the Čech closure spaces, which satisfy (K0), (K1), (K2), and (K3), are identical to the pretopological spaces which also form a topological category, see e.g. [29].

4. Symmetry and Separation Axioms

The symmetry and separation axiom always deal with the existence of neighborhoods around every point of X such that certain additional conditions are satisfied. Hence we shall assume $\mathcal{N}(x) \neq \emptyset$, i.e. (K0') from now on.

4.1. Symmetry Axioms. The two most important symmetry properties in term of neighborhoods and their closure counterparts are

(R0) If x is contained in each neighborhood of y then y is contained in each neighborhood of x.

(R0c) If $x \in \mathsf{cl}(\{y\})$ then $y \in \mathsf{cl}(\{x\})$.

(wS) If x is contained in each neighborhood of y then $\mathcal{N}(x) \subseteq \mathcal{N}(y)$.

(cS) $\mathsf{cl}(A) \cap \mathsf{cl}(\{x\}) \neq \emptyset$ implies $x \in \mathsf{cl}(A)$.

(S) If x is contained in each neighborhood of y then $\mathcal{N}(x) = \mathcal{N}(y)$.

(RE) If $N_x \cap N_y \neq \emptyset$ for all $N_x \in \mathcal{N}(x)$ and all $N_y \in \mathcal{N}(y)$ implies $\mathcal{N}(x) = \mathcal{N}(y)$

The (R0) axiom was introduced by Šanin [30]. Eduard Čech proved that a pretopological space is semi-uniformizable if and only if it satisfies (R0c) [5, Thm.23.B.3]. In [23] it is shown that (R0) is equivalent to "weak uniformizability".

In [5, Sect.29] axiom (S) and its variants appear as necessary condition for normality. The equivalence of (wS), (cS) and (S) for preotopological space is shown as Thm.29.A.3.

Reciprocal spaces were considered in [21], where (RE) was termed "axiom P".

Lemma 5. *(i) If (X, cl) is an extended topology then (R0) and (R0c) is equivalent.*
(ii) If (X, cl) is a neighborhood space then (wS) implies (R0).
(iii) If (X, cl) is a neighborhood space then (RE) always implies (cS).

Proof. (i) can be seen as follows: Let $N(x) = \cap\{N | N \in \mathcal{N}(x)\}$ be the vicinity of x. Then (R0) can be written as "$y \in N(x)$ implies $x \in N(y)$. The crucial observation is that in isotonic space we have $\mathsf{cl}(\{x\}) = \{y | \forall N \in \mathcal{N}(y) : N \cap \{x\} \neq \emptyset\} = \{y | x \in N(x)\}$, i.e.,

$$x \in N(y) \Longleftrightarrow y \in \mathsf{cl}(\{x\}). \tag{16}$$

The proof now follows trivially.
(ii) Suppose $x \in N(x)$. Then (wS) implies $\mathcal{N}(x) \subseteq \mathcal{N}(y)$, i.e., $U \in \mathcal{N}(x) \implies U \in \mathcal{N}(y)$. Equivalently, $x \notin \mathsf{cl}(-U)$ implies $y \notin \mathsf{cl}(-U)$ and hence $y \in \mathsf{cl}(A)$ implies $x \in \mathsf{cl}(A)$ for all A. Now $x \in N(y)$, i.e., $y \in \mathsf{cl}(\{x\})$, and, by (K2), $y \in \mathsf{cl}(\{y\})$, implies $x \in \mathsf{cl}(\{y\})$, i.e., $y \in N(x)$.
(iii) If $y \in N(x)$ and by (K2) $y \in N(y)$ then $N(x) \cap N(y) \neq \emptyset$ and hence $N_x \cap N_y \neq \emptyset$ for all $N_x \in \mathcal{N}(x)$ and all $N_y \in \mathcal{N}(y)$. Now (RE) implies $\mathcal{N}(x) = \mathcal{N}(y)$, i.e., (S) holds as well. $\square$

Theorem 12. *(i) (S) implies (cS) implies (wS) in isotonic spaces.*
(ii) In an isotonic (R0) space (wS), (cS), and (S) are equivalent.
(iii) If (X, cl) is a neighborhood space then (wS), (cS), and (S) are equivalent.

Proof. (i) Suppose (S) holds, i.e., $x \in N(y) \Longleftrightarrow y \in \mathsf{cl}(\{x\})$ implies that $N \in \mathcal{N}(x)$ iff $N \in \mathcal{N}(y)$. Equivalently, we have $x \notin \mathsf{cl}(-N)$ iff $y \notin \mathsf{cl}(-N)$, i.e., $x \in \mathsf{cl}(-N)$ iff $y \in \mathsf{cl}(-N)$, i.e., $x \in \mathsf{cl}(A)$ iff $y \in \mathsf{cl}(A)$ for all $A \in \mathcal{P}(X)$. Thus $y \in \mathsf{cl}(\{x\})$ and $y \in \mathsf{cl}(A)$ implies $x \in \mathsf{cl}(A)$, i.e., $\mathsf{cl}(\{x\}) \cap \mathsf{cl}(A) \neq \emptyset$ implies $x \in \mathsf{cl}(A)$. The negation of this implication is (cS).

Now let $x \in N(y)$, i.e., $y \in \mathsf{cl}(\{x\})$ and $U \in \mathcal{N}(x)$, i.e., $x \notin \mathsf{cl}(-U)$. (cS) implies $\mathsf{cl}(\{x\}) \cap \mathsf{cl}(-U) = \emptyset$, hence $y \in \mathsf{cl}(\{x\})$ implies $y \notin \mathsf{cl}(-U)$, i.e. $y \in \mathsf{int}(U)$, and hence $U \in \mathcal{N}(y)$. Thus $\mathcal{N}(x) \subseteq \mathcal{N}(y)$.

(ii) It suffices to show that (R0) and (wS) implies (S). We have $x \in \mathsf{cl}(\{y\})$ implies $\mathcal{N}(y) \subseteq \mathcal{N}(x)$ by (wS) and $y \in \mathsf{cl}(\{x\})$ by (R0). Applying (wS) again we have $\mathcal{N}(x) \subseteq \mathcal{N}(y)$, and the assertion follows.

(iii) Follows directly from (ii) and Lemma 5(ii). □

It is well known that (R0) and (S) are equivalent in topological spaces, see e.g. [29].

4.2. Lower Separation Axioms. The axioms (T0) and (T1), as well as the weak symmetry axiom (R0) above, can be stated in full generality in the same way as in topological spaces:

(T0) For all $x, y \in X$, $x \neq y$ there is $N' \in \mathcal{N}(x)$ such that $y \notin N'$ or there is $N'' \in \mathcal{N}(y)$ such that $x \notin N''$.

(T1) For all $x, y \in X$, $x \neq y$ there is $N' \in \mathcal{N}(x)$ and $N'' \in \mathcal{N}(y)$ such that $x \notin N''$ and $y \notin N'$.

Assuming (K0') we can rewrite (T0) and (T1) in terms of $N(x)$. This notation immediately suggests corresponding axioms in terms of the closure function:

(T0) $x \neq y$: $y \notin N(x)$ or $x \notin N(y)$ (cT0) $x \neq y$: $y \notin \mathsf{cl}(\{x\})$ or $x \notin \mathsf{cl}(\{y\})$

(T1) $N(x) \subseteq \{x\}$ (cT1) $\mathsf{cl}(x) \subseteq \{x\}$

Theorem 13. *(T1)$\Longleftrightarrow$(T0 and R0) and (cT1)$\Longleftrightarrow$(cT0 and cR0). If (X, cl) is an extended topology then (T0)$\Longleftrightarrow$(cT0) and thus also (T1)$\Longleftrightarrow$(cT1).*

Proof. It is obvious from the definitions that (T1) $\Longrightarrow$ (T0) and (R0) and that (cT1) $\Longrightarrow$ (cT0) and (cR0). Conversely, suppose (X, cl) satisfies (T0) and (R0) but not (T1). Then there are two points $x_0 \neq y_0 \in X$ such that $y_0 \in N(x_0)$. Now (T0) implies $x_0 \notin N(y_0)$, contradicting (R0). The argument for (cT1) is analogous. The second part of theorem follow immediately from Lemma 5 and equ.(16). □

4.3. Hausdorff-Like Axioms.

(T2) If $x \neq y$ then there is $N' \in \mathcal{N}(x)$ and $N'' \in \mathcal{N}(y)$ such that $N' \cap N'' = \emptyset$.

(H) A proper prefilter converges to at most one point.

(T2½) If $x \neq y$ then there is $N' \in \mathcal{N}(x)$ and $N'' \in \mathcal{N}(y)$ such that $\mathsf{cl}(N') \cap \mathsf{cl}(N'') = \emptyset$.

Lemma 6. *(T2) implies (RE) in general.*
In a neighborhood space (T2½) $\Longrightarrow$ (T2)$\Longleftrightarrow$(T0 and RE) $\Longrightarrow$ (T1).

Proof. (T2) $\Longrightarrow$ (RE): $N' \cap N'' \neq \emptyset$ for all $N' \in \mathcal{N}(x)$ and all $N'' \in \mathcal{N}(y)$ implies $x = y$ and hence trivially $\mathcal{N}(x) = \mathcal{N}(y)$.

Now assume that (X, cl) is a neighborhood space.

(T2$\frac{1}{2}$) $\implies$ (T2) $\implies$ (T1): If $\mathsf{cl}(N') \cap \mathsf{cl}(N'') = \emptyset$ then (K2) implies $N' \cap N'' = \emptyset$ and using (K2) again implies both $x \notin N''$ and $y \notin N'$.

Now suppose (RE) holds but (T2) is not satisfied. Then there are points $x \neq y$ such that $N' \cap N'' \neq \emptyset$ for all $N' \in \mathcal{N}(x)$ and all $N'' \in \mathcal{N}(y)$. By (RE) we have $\mathcal{N}(x) = \mathcal{N}(y)$. (K2) now implies $x, y \in N$ for all $N \in \mathcal{N}(x) = \mathcal{N}(y)$. This contradicts (T1) and hence also (T0). $\qquad\square$

Theorem 14. *If (X, cl) is a neighborhood space then (H)$\Longleftrightarrow$(T2).*

Proof. In a neighborhood space each neighborhood system $\mathcal{N}(x)$ is by definition a proper prefilter. The union $\mathcal{F} \cup \mathcal{G}$ of two proper prefilters is a proper prefilter if and only if $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$ and all $G \in \mathcal{G}$.

(T2) $\implies$ (H): Suppose $\mathcal{F}$ is a proper prefilter converging to both x and y, $\mathcal{F} \to x$ and $\mathcal{F} \to y$. In other words, $\mathcal{N}(x) \cup \mathcal{N}(y) \subseteq \mathcal{F}$. If $x \neq y$ then by (T2) there is $N' \in \mathcal{N}(x)$ and $N'' \in \mathcal{N}(y)$ such that $N' \cap N'' = \emptyset$, i.e., $\mathcal{F}$ is not a proper prefilter. Thus $x = y$.

(H) $\implies$ (T2): Condition (H) can be rephrased in as "if $x \neq y$ then there is no proper prefilter finer than $\mathcal{N}(x) \cup \mathcal{N}(y)$, and, equivalently, $\mathcal{N}(x) \cup \mathcal{N}(y)$ itself is not a proper prefilter. This means that there is $N', N'' \in \mathcal{N}(x) \cup \mathcal{N}(y)$ such that $N' \cap N'' = \emptyset$. Since both $\mathcal{N}(x)$ and $\mathcal{N}(y)$ are proper prefilters we have to choose $N' \in \mathcal{N}(x)$ and $N'' \in \mathcal{N}(y)$, and hence (T2) holds. $\qquad\square$

4.4. Regularity-Like Axioms.

Theorem 15. *In an isotonic space the following conditions are equivalent:*

(R) *For all $x \in X$ and all $N \in \mathcal{N}(x)$ there is $U \in \mathcal{N}(x)$ such that $\mathsf{cl}(U) \subseteq N$.*

(R') *For all $x \in X$ and all non-empty $A \in \mathcal{P}(X)$ such that $x \notin \mathsf{cl}(A)$ there is $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(A)$ such that $U \cap V = \emptyset$.*

Proof. Suppose (R') holds. Choose an arbitrary $x \in X$ and $N \in \mathcal{N}(x)$ and set $A = -N$. We have $x \notin \mathsf{cl}(A)$ iff $-A = N \in \mathcal{N}(x)$. Now We have $U \cap V = \emptyset$, hence $U \subseteq -V$ and by isotony $\mathsf{cl}(U) \subseteq \mathsf{cl}(-V)$ and finally $\mathsf{int}(V) = -\mathsf{cl}(-V) \subseteq -\mathsf{cl}(U)$. By Lemma 1 we have $A \subseteq \mathsf{int}(V)$, thus $A \subseteq -\mathsf{cl}(U)$, and $\mathsf{cl}(U) \subseteq -A = N$. Thus (R) is satisfied.

Conversely assume (R), and let $N \in \mathcal{N}(x)$. Then there is $U \in \mathcal{N}(x)$ such that $\mathsf{cl}(U) \subseteq N$. Set $V = -U$ and $A = -N$. Then $-N = A \subseteq -\mathsf{cl}(U) = -\mathsf{cl}(-V) = \mathsf{int}(V)$, i.e., $A \in \mathcal{N}(V)$ and $U \cap V = \emptyset$. Observing again that $N \in \mathcal{N}(x)$ if and only if $x \notin \mathsf{cl}(A)$ completes the proof. $\qquad\square$

Definition 11. *An isotonic space is* regular *if it satisfies one of conditions in theorem 15.*

It is worth noting that condition (R) naturally appears in the theory of generalized convergence spaces (see e.g. []), while (R') is the straightforward generalization of the usual regularity axiom in topological spaces.

A stronger property is

(tR) For all $x \in X$ and all non-empty closed sets $\emptyset \neq A = \mathsf{cl}(A) \in \mathcal{P}(X)$ there is $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(A)$ such that $U \cap V = \emptyset$.

It is clear that (tR) implies (R). Obviously (R) and (tR) are equivalent if cl is idempotent.

Definition 12. *A closure spaces is (T3) if it satisfies (R) and (T0).*

Lemma 7. *If (X, cl) is a neighborhood space then (T3) $\Longrightarrow$ (T2$\frac{1}{2}$) $\Longrightarrow$ (T2).*

Proof. We start with (R') and set $A = \{y\}$. Since (R) implies (R0) we know that a (T3) space is (T1), hence $\mathsf{cl}(A) = \mathsf{cl}(\{y\}) = \{y\}$. Thus (R') reduces to the existence of $N' \in \mathcal{N}(x)$ and $N'' \in \mathcal{N}(y)$ such that $N' \cap N'' = \emptyset$, i.e., to (T2). Now we can use (R) to obtain $U' \in \mathcal{N}(x)$ and $U'' \in \mathcal{N}(y)$ with $\mathsf{cl}(U') \subseteq N'$ and $\mathsf{cl}(U'') \subseteq N''$. Clearly $\mathsf{cl}(U') \cap \mathsf{cl}(U'') = \emptyset$, i.e., (T2$\frac{1}{2}$) is satisfied. $\square$

4.5. Normality. There appears to be no consistent definition of normal spaces beyond the realm of topological spaces. The nomenclature above is a hybrid of the conventions from different authors, motivated in part by analogy with the terminology of regularity properties above. Cech [5,] call a pretopological space normal if it is quasi-normal and symmetric. In [27] Paoli & Ripoli introduce a notion of normal convergence space based on closed sets, calling (QTN) quasi-normal and using normal for (N). In [34] (QN) is called normal.

Definition 13. *Let (X, cl) be a isotonic space. The space is*

(QN) quasi-normal *if for all $A, B \neq \emptyset$ $\mathsf{cl}(A) \cap \mathsf{cl}(B) = \emptyset$ implies that there is $U \in \mathcal{N}(A)$ and $V \in \mathcal{N}(B)$ such that $U \cap V = \emptyset$.*

(N) normal *For all non-empty closed sets $A = \mathsf{cl}(A) = \neq \emptyset$ and all $N \in \mathcal{N}(A)$ there is $U \in \mathcal{N}(A)$ such that $\mathsf{cl}(U) \subseteq N$.*

(TN) normal *For all non-empty closed sets $A = \mathsf{cl}(A) = \neq \emptyset$ and all $N \in \mathcal{N}(A)$ there is a closed set $U = \mathsf{cl}(U) \in \mathcal{N}(A)$ such that $\mathsf{cl}(U) \subseteq N$.*

(QTN) quasi-t-normal *if for all non-empty disjoint closed sets, i.e., $A = \mathsf{cl}(A) \neq \emptyset$, $B = \mathsf{cl}(B) \neq \emptyset$, $A \cap B \neq \emptyset$ implies that there is $U \in \mathcal{N}(A)$ and $V \in \mathcal{N}(B)$ such that $U \cap V = \emptyset$.*

Lemma 8. *Let (X, cl) be a neighborhood space. Then*
(i) (QN) implies (N) implies (QTN), and (TN) implies (N).
(ii) If cl is idempotent then (QN), (TN), (N), and (QTN) are equivalent.

Proof. (i) The implication (TN) $\Longrightarrow$ (N) is trivial.
Suppose $Q = \mathsf{cl}(Q) \neq \emptyset$ and $W \in \mathcal{N}(Q)$, i.e., $Q \in \mathsf{int}(W)$. Using $Q = \mathsf{cl}(Q)$ and $-\mathsf{int}(W) = \mathsf{cl}(-W)$ we have $\mathsf{cl}(Q) \cap \mathsf{cl}(-W) = \emptyset$ and by (QN) there is $U \in \mathcal{N}(Q)$ and $V \in -\mathcal{W}$ such that $U \cap V = \emptyset$. Thus $(-W) \subseteq \mathsf{int}(V) = -\mathsf{cl}(-V)$ and hence $\mathsf{cl}(-V) \subseteq W$. Furthermore $U \subseteq -V$. Now $U \in \mathcal{N}(Q)$ and $U \subseteq -V$ implies $-V \in \mathcal{N}(Q)$ and $\mathsf{cl}(-V) \subseteq W$; thus (N) holds.
(second part of (i) missing).
(ii) The equivalence of (N) and (TN) and the equivalence of (QN) and (QTN), respectively is obvious for idempotent closure operators. Then lemma now follows from (i). $\square$

Definition 14. *A closure space* (X, cl) *is (T4) if satisfies (T1) and (QN).*

Lemma 9. *If* (X, cl) *is a neighborhood space, then (T4) implies (T3).*

Proof. Consider a point x and a set A such that $x \notin \mathsf{cl}(A)$. From (T1) and (K2) we have $\mathsf{cl}(\{x\}) = \{x\}$, hence $\mathsf{cl}(\{x\}) \cap \mathsf{cl}(A) = \emptyset$. Thus (QN) implies that there is $N \in \mathcal{N}(x)$ and $U \in \mathcal{N}(A)$ such that $N \cap U = \emptyset$, i.e., (R') is satisfied. Recall, finally, that (R') and (T0) is equivalent to (T3) in all isotonic spaces. $\qquad\square$

Cech [5] mentions that a symmetric, quasi-normal pretopological space is topological.

5. Connectedness

Connectedness is closely related to separation. Two sets $A, B \in \mathcal{P}(X)$ are *semi-separated* if there are neighborhoods $N' \in \mathcal{N}(A)$ and $N'' \in \mathcal{N}(B)$ such that $A \cap N'' = N' \cap B = \emptyset$; they are *separated* if if there are neighborhoods $N' \in \mathcal{N}(A)$ and $N'' \in \mathcal{N}(B)$ such that $N' \cap N'' = \emptyset$.

Lemma 10. *If* (X, cl) *is isotone then A and B are semi-separated if and only if* $\mathsf{cl}(A) \cap B = A \cap \mathsf{cl}(B) = \emptyset$.

Proof. Using the definition of the neighborhoods of sets we see that A and B are semi-separated if and only if there are sets U and V such that $A \subset \mathsf{int}(U)$ and $B \subset \mathsf{int}(V)$. We argue: $A \cap V = \emptyset$ implies $V \in -A$ and hence $B \subseteq \mathsf{int}(V) \subseteq \mathsf{int}(-A)$ where we have used that cl is isotone. Switching to the complements we have $\mathsf{cl}(A) = A - \mathsf{int}(-A) \subseteq -B$ and hence $\mathsf{cl}(A) \cap B = \emptyset$. Analogously we see $\mathsf{cl}(B) \subseteq -A$.
Now suppose $\mathsf{cl}(A) \cap B = A \cap \mathsf{cl}(B) = \emptyset$. Thus $B \subseteq -\mathsf{cl}(A) = \mathsf{int}(-A)$ and hence $V = -A \in \mathcal{N}(B)$. Of course $A \cap V = \emptyset$. Analogously we see that $U = -B$ is a neighborhood of A. $\qquad\square$

Definition 15. *A set $Z \in \mathcal{P}(X)$ is* connected *in (X, cl) if it is not a disjoint union of a nontrivial semi-separated pairs of sets $A, Z - A$, $A \neq \emptyset, Z$.*

By definition a 1-point set $Z = \{z\}$ is connected. We say that (X, cl) is connected if X is connected in (X, cl). This definition can be rephrased as

Theorem 16. *A set $Z \in \mathcal{P}(X)$ is* connected *in an isotonic space (X, cl) if and only if for each proper subset $A \subseteq Z$ holds*

$$[\mathsf{cl}(A) \cap (Z \setminus A)] \cup [\mathsf{cl}(Z \setminus A) \cap A] \neq \emptyset \tag{17}$$

Eq.(17) is known as the Hausdorff-Lennes condition.

Lemma 11. *If X and Y are connected in an isotone space (X, cl) and $X \cap Y \neq \emptyset$, then $X \cup Y$ is connected.*

Proof. We use the Hausdorff-Lennes condition:
$$
\begin{aligned}
&[\mathsf{cl}(A) \cap (Y \cup Z) \setminus A] \cup [A \cap \mathsf{cl}((Y \cup Z) \setminus A)] = \\
&[\mathsf{cl}(A) \cap (Y \setminus A)] \cup [\mathsf{cl}(A) \cap (Z \setminus A)] \cup [A \cap \mathsf{cl}((Y \setminus A) \cup (Z \setminus A))] \supseteq \\
&\{[\mathsf{cl}(A) \cap (Y \setminus A)] \cup [A \cap \mathsf{cl}(Y \setminus A)]\} \cup \{[A \cap \mathsf{cl}(Z \setminus A)] \cup [\mathsf{cl}(A) \cap (Z \setminus A)]\}
\end{aligned}
\tag{18}
$$

If $A \cap Y$ or $A \cap Z$ is a proper subset of Y or Z, respectively, then one of the expressions in the braces in non-empty. Both expressions are empty if and only if either $A = Z$ and $A \cap Y = \emptyset$ or $A \cap Z = \emptyset$ and $A = Y$. This is impossible if $Y \cap Z \neq \emptyset$. $\square$

Theorem 17. *If (X, cl) is a neighborhood space then $\mathsf{cl}(Z)$ is connected whenever Z is connected.*

Proof. Set $A' = Z \cap A$ and $A'' = A \setminus Z$. We then use the Hausdorff-Lennes condition:

$$\begin{aligned} [\mathsf{cl}(A) \cap (\mathsf{cl}(Z) \setminus A)] \cup [\mathsf{cl}(\mathsf{cl}(Z) \setminus A) \cap A] &\supseteq \\ [(\mathsf{cl}(A') \cup \mathsf{cl}(A'')) \cap (Z \setminus A')] \cup [\mathsf{cl}(Z - A') \cap (A' \cup A'')] &\supseteq \quad (19)\\ \{[\mathsf{cl}(A') \cap (Z \setminus A')] \cup [\mathsf{cl}(Z \setminus A') \cap A']\} [\mathsf{cl}(Z - A') \cap A''] \end{aligned}$$

Here we have used $Z - A' \subseteq (\mathsf{cl}(Z) \setminus A'') \setminus A'$ which is true only if (K2) holds. If $A' \neq \emptyset$ then the term in braces in non-empty because Z is connected by assumption. If $A' = \emptyset$ then $A'' \subseteq \mathsf{cl}(Z) \setminus Z$ is nonempty and hence $\mathsf{cl}(Z \setminus A') \cap A'' = \mathsf{cl}(Z) \cap A'' \neq \emptyset$. Thus $\mathsf{cl}(Z)$ is connected. $\square$

Lemma 12. *In neighborhood spaces let $f : (X, \mathsf{cl}) \to (Y, \mathsf{cl})$ be continuous and suppose $A, B \subseteq X$ are semi-separated. Then $f^{-1}(A)$ and $f^{-1}(A)$ are semi-separated.*

Proof. Suppose $\mathsf{cl}(A) \cap B = \emptyset$. Then $f^{-1}(\mathsf{cl}(A)) \cap f^{-1}(B) = \emptyset$. Continuity means $\mathsf{cl}(f^{-A}) \subseteq f^{-1}(\mathsf{cl}(A))$, hence $\mathsf{cl}(f^{-1}(A)) \cap f^{-1}(B) = \emptyset$. By the same argument $\mathsf{cl}(B) \cap A = \emptyset$ implies $\mathsf{cl}(f^{-1}(B)) \cap f^{-1}(A) = \emptyset$, and the lemma follows. $\square$

Theorem 18. *If $f : (X, \mathsf{cl}) \to (Y, \mathsf{cl})$ is a continuous function between neighborhood spaces, and A is connected in X, then $f(A)$ is connected in Y.*

Proof. Suppose $f(A)$ is not connected. Then $f(A) = U \dot\cup V$, where U and V are semiseparated and non-empty. Thus $f^{-1}(U)$ and $f^{-1}(V)$ are semi-separated. Clearly $A' = A \cap f^{-1}(U)$ and $A'' = A \cap f^{-1}(V)$ are both non-empty and also semi-separated. Furthermore $A' \cup A'' = A$, hence A is not connected. $\square$

Appendix A: Filters and Grills

Let $\mathcal{F}$ be an arbitrary collection of subsets of X. The *dual* of $\mathcal{F}$ is defined by

$$\mathcal{F}^* = \{F \in \mathcal{P}(X) | - F \notin \mathcal{F}\} \quad (20)$$

Obviously we have $\mathcal{F}^{**} = \mathcal{F}$, $\mathcal{F} = \mathcal{G}^*$ if and only if $\mathcal{G} = \mathcal{F}^*$, and $\mathcal{P}(X)^* = \varnothing$.

A *stack* on X is $\mathcal{F}$ a (possibly empty) isotonic collection of subsets of X. $\mathcal{F}$ is a stack if and only if its dual $\mathcal{F}^*$ is a stack. In particular, $\mathcal{P}(X)^*$ and $\varnothing$ are stacks on X.

We call a stack proper if it is non-empty and does not contain the empty set. Note that the only stack containing $\emptyset$ is $\mathcal{P}(X)$ as an immediate consequence of isotony. Thus the dual of a proper stack is again a proper stack. Equivalently, a stack $\mathcal{F}$ is proper if and only if $X \in \mathcal{F}$. In table 3 we list the filter axioms and their duals.

Proper stacks satisfy (F0). The "pair-wise intersection" axiom (F1) is emphasized in [35], the term *prefilter* for an (F1)-stack was used in [1]. Axiom (F2), which implues (F1), defines H. Cartan's notion of a *filter* [4], its dual is called a *grill* [7]. By analogy

Table 3. Filter and Grill Axioms.

Axiom	filter	grill
ISO	$F \in \mathcal{F}, F \subset F' \implies F' \in \mathcal{F}$	
F0	$\mathcal{F} \neq \varnothing$ and $\emptyset \notin \mathcal{F}$, i.e., $X \in \mathcal{F}$	
F1	$F', F'' \in \mathcal{F} \implies F' \cap F'' \neq \emptyset$ $F \in \mathcal{F} \implies -F \notin \mathcal{F}$	$G' \cup G'' = X \implies G' \in \mathcal{F}$ or $G' \in \mathcal{F}$ $G \notin \mathcal{F} \implies -G \in \mathcal{F}$
F2	$F', F'' \in \mathcal{F} \implies F' \cap F'' \in \mathcal{F}$	$G' \cup G'' \in \mathcal{F} \implies G' \in \mathcal{F}$ or $G' \in \mathcal{F}$
F3	$F \in \mathcal{F} \iff -F \notin \mathcal{F}$	

we call a stack that satisfies the dual of (F1) a *pregrill*. A stack with property (F3) is *universal*. A universal filter is an *ultrafilter* [3]. Obviously (F3) implies (F1) but not (F2).

Notation

x Points are denotes by lower case letters, preferably x, y, z, p, q, ...

X A set of points is denoted by captial letters.

$\mathcal{P}(X)$ The power set if X.

$\mathcal{F}$ A family of sets, e.g., $\mathcal{F} \subseteq \mathcal{P}(X)$.

$\mathcal{N}(x)$ The familly of neighborhoods of x.

$\mathcal{F}^*$ The dual of $\mathcal{F}$.

$\mathfrak{P}(X) = \mathcal{P}(\mathcal{P}(X))$, the set of all families of sets on X.

$\mathfrak{S}(X)$ The set of all stacks on X.

$\mathfrak{S}_0(x)$ The set of all proper stacks on X.

$\mathfrak{J}_0(x)$ The set of all proper prefiltres on X.

$\mathfrak{F}(x)$ The set of all filters on x.

$\mathfrak{F}_0(x)$ The set of all proper filters on x.

References

[1] M. M. Brissaud. Mathématiques économiques: Agregation des préférences individuelles. *C. R. Acad. Sc. Paris Ser. A*, 278:637–639, 1974.

[2] M. M. Brissaud. Les espaces prétopologiques. *C. R. Acad. Sc. Paris Ser. A*, 280:705–708, 1975.

[3] H. Cartan. Filtres et ultrafiltres. *C. R. Acad. Sci., Paris*, 205:777–779, 1937.

[4] H. Cartan. Theorie des filtres. *C. R. Acad. Sci., Paris*, 205:595–598, 1937.

[5] E. Čech. *Topological Spaces*. Wiley, London, 1966.

[6] G. Choquet. Convergences. *Ann. Univ. Grenoble*, 23:55–112, 1947.

[7] G. Choquet. Sur les notions de filtres et de grille. *C. R. Acad. Sc. Paris Ser. A*, 224:171–173, 1947.

[8] M. M. Day. Convergence, closure, and neighborhoods. *Duke Math. J.*, 11:181–199, 1944.

[9] D. Dikranjan, E. Guili, and A. Tozzi. Topological categories and closure operators. *Quest. Math.*, 11:323–337, 1988.

[10] H. R. Fischer. Limesräume. *Math. Annalen*, 137:269–303, 1959.

[11] A. Galton. Continuous motion in discrete space. In A. G. Cohn, F. Giunchiglia, and B. Selman, editors, *Principles of Knowledge Representation and Reasoning: Proceedings of the Seventh International Conference (KR2000)*, pages 26–37, San Francisco, CA, 2000. Morgan Kaufmann Publishers.

[12] G. C. Gastl and P. C. Hammer. Extended topology. Neighborhoods and convergents. In N.N., editor, *Proceedings of the Colloquium on Convexity 1965*, pages 104–116, Copenhagen, DK, 1967. Københavns Univ. Matematiske Inst.

[13] S. Gniłka. On extended topologies. I: Closure operators. *Ann. Soc. Math. Pol., Ser. I, Commentat. Math.*, 34:81–94, 1994.

[14] S. Gniłka. On extended topologies. II: Compactness, quasi-metrizability, symmetry. *Ann. Soc. Math. Pol., Ser. I, Commentat. Math.*, 35:147–162, 1995.

[15] S. Gniłka. On continuity in extended topologies. *Ann. Soc. Math. Pol., Ser. I, Commentat. Math.*, 37:99–108, 1997.

[16] P. Hamburg. Konvergenz in verallgemeinerten topologischen Räumen. *Port. Math.*, 35:137–168, 1976.

[17] P. C. Hammer. General topoloy, symmetry, and convexity. *Trans. Wisconsin Acad. Sci., Arts, Letters*, 44:221–255, 1955.

[18] P. C. Hammer. Extended topology: Set-valued set functions. *Nieuw Arch. Wisk. III*, 10:55–77, 1962.

[19] P. C. Hammer. Extended topology: Continuity I. *Portug. Math.*, 25:77–93, 1964.

[20] P. C. Hammer. Extended topology and systems. *Math. Systems Theory*, 1:135–142, 1967.

[21] H. H. Keller. Die Limes-Uniformisierbarkeit der Limesräume. *Math. Ann.*, 176:334–341, 1968.

[22] D. C. Kent. Convergence functions and their related topologies. *Fund. Math.*, 54:125–133, 1964.

[23] D. C. Kent. On convergence groups and convergence uniformities. *Fund. Math.*, 60:213–222, 1967.

[24] H. L. Kowalsky. Limesräume und komplettierung. *Math. Nachr.*, 12:301–340, 1954.

[25] E. H. Moore and H. L. Smith. A general theory of limits. *Amer. J. Math.*, 44:102–121, 1922.

[26] J. Novák. On some problems concerning multivalued convergences. *Czech. Math. J.*, 14:548–561, 1964.

[27] M. Paoli and E. Ripoli. Convergence subspaces and normality. *Ital. J. Pure Appl. Math.*, 1:91–99, 1997.

[28] G. Preuss. *Theory of Topological Structures*. D. Reidel, Dordrecht, NL, 1988.

[29] G. Preuß. Convenient topology. *Math. Japonica*, 47:171–183, 1998.

[30] N. Šanin. On separation in topological space. *Dok. Akad. Nauk SSSR*, 38:110–113, 1943.

[31] W. Sierpiński. *General Topolgy*. Univ. Toronto Press, Toronto, 1956.

[32] M. B. Smyth. Semi-metric, closure spaces and digital topology. *Theor. Computer Sci.*, 151:257–276, 1995.

[33] W. P. Soltan. *An Introduction in Axiomatic Theory of Convexity*. Shtiintsa, Kishinev, 1984. Russian.

[34] D. Thampuran. Normal neighborhood spaces. *Rend. Sem. Mat. Univ. Padova*, 45:95–97, 1971.

[35] D. V. Thampuran. Extended topology: filters and convergence I. *Math. Annalen*, 158:57–68, 1965.

[36] D. V. Thampuran. Extended topology: Moore-smith convergence. *Portug. Math.*, 24:83–98, 1965.

[37] D. V. Thampuran. Neighborhood spaces and convergence. *Port. Math.*, 33:43–49, 1974.