Recombination Spaces, Metrics, and Pretopologies

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Abstract

The topological features of genotype spaces given a genetic operator have a substantial impact on the course of evolution. We explore the structure of the recombination spaces arising from four different unequal crossover models in the context of pretopological spaces. We show that all four models are incompatible with metric distance measures due to a lack of symmetry.

Keywords: Unequal Crossover, Recombination Spaces, Pretopology

1. Introduction

In a recent paper it was suggested that the topology induced by unequal crossover is incompatible with the existence of a metric that reflects the accessibility relationships among chromosomes with different numbers of gene copies [25]. In this contribution we provide a simple but rigorous proof of this suggestion for four types of models. The one is the model analyzed in Shpak and Wagner [25] which allows for any unequal crossover with the same probability. The second model is the one analyzed by Kruger and Vogel [17] where copy number on a chromosome can only change one copy per unequal crossover event. The unequal sister chromatide exchange model [4] can be viewed as restriction of the first two models. The forth case is an extension of the Shpak-Wagner model that that distinguishes the individual duplicate genes.

The significance of the non-metrizability results is that there are plausible genetic mechanisms, like unequal crossover, which can not be modeled as a process on a metric space. In other words there is no natural notion of similarity that has the formal properties of a metric and that reflects the actual genetic accessibility relationships. This is in contrast to classical models of, for instance, quantitative polygenic variation, which occurs in a Euclidian space. Together with independent evidence from RNA secondary structure evolution [8, 10, 11], these results show that evolutionary dynamics may occur on exotic topological structures rather than the more familiar metric spaces. For a discussion of the potential implications see [25]. It is argued at length in [27] that pretopological spaces provide a framework with the appropriate level of generality.

In classical topology a topological space is called metric if there exists a distance function such that the neighborhood basis of every point x is determined by the collection of all balls with radius $\varepsilon > 0$ centered at x. In particular, in the case of finite sets this implies that $\{x\}$ itself is a neighborhood and hence the metric induces the trivial discrete topology. On the other hand, the Hamming graph is associated with a metric in a natural way, namely $d(x, y)$ is the number of point mutations separating two sequences x and y. From this we conclude that the concept of metrization needs to be generalized to be useful in the context of genotype or phenotype spaces, which are finite or at worst countable. Here we provide such a generalization and use it to prove the suggestion about the non-existence of a metric on the unequal recombination space.

This contribution is organized as follows: In the following three sections we outline the relationships between pretopological neighborhood structures and metric distance measures, focussing on separation and regularity properties. In section 7 we examine the neighborhood structures introduced by four models of unequal crossover. In order to make this manuscript self-contained we add three appendices containing the basic facts about pretopological spaces, metrics, and the proofs of all theorems stated in the following three sections. While most of this material is not new, there appears to be no convenient reference to elementary proofs.

2. Metrics and Pretopologies

The relationships of metric distance measures and neighborhood systems is one of the central themes of set-theoretic topology [18]. Metric distance measures also play a prominent role in graph theory [6]. The use of metrics in topology and graph theory at first glance appears to be unrelated, since graphs in general cannot be regarded as topological spaces.

The relation of metrics and topologies is based on the notion of ε-balls. Let $d : X \times X \to \mathbb{R}_0^+$ be a metric on X. We call the sets of the form

$$\begin{aligned} B_\varepsilon(x) &= \{y \in X | d(x,y) < \varepsilon\} \\ B'_\varepsilon(x) &= \{y \in X | d(x,y) \leq \varepsilon\} \end{aligned} \tag{1}$$

ε-balls with and without border centered at x, respectively. Given a metric d on X the topology induced by d is defined by the neighborhood basis[1]

$$\mathcal{B}(x) = \{B_\varepsilon(x) | \varepsilon > 0\} \tag{2}$$

for each $x \in X$. As an immediate consequence, if X is a finite set then there is a minimum distance d^* between any two distinct points and hence $B_\varepsilon(x) = \{x\}$ for $0 < \varepsilon < d^*$. Thus metrics on finite spaces, including graphs, give rise only to the trivial *discrete* topology.

On the other hand, there is a natural metric associated with every undirected graph $\Gamma(V, E)$ with vertex set V and edge set E. The canonical distance $d_\Gamma(x, y)$ is defined as the minimum number of edges in any path that connects x with y. There is one-to-one correspondence between finite pretopological spaces $(X, \mathcal{N})$ and directed graphs $\Gamma(X, E)$ with vertex set X by means of the following simple construction: For each $x \in X$ define a *directed edge* (x, y) "from x to y" if and only if y is contained in the intersections of all neighborhoods (i.e., in the smallest neighborhood) of x.

Pretopological spaces, as a generalization of both finite simple graphs and topological spaces provide a means of unifying these two seemingly distinct uses of metrics. A pretopological space is conveniently defined by specifying (a basis of) its neighborhood filter $\mathcal{N}(x)$, or by means of a closure operator. Pretopologies are generalizations of topological spaces that lack the most important property of topologies: Neighborhoods do not contain *open* neighborhoods here. Pretopologies provide a meaningful theory of neighborhood systems without recourse to open sets, which do not seem to arise naturally in the context of either genotype or phenotype spaces [27]. The details of the axiom systems for pretopological spaces and the relationships of pretopologies to the more familiar topological spaces are collected in Appendix A.

[1] In order to make this contribution self-contained we give definitions of the most important concepts of set-theoretic topology in the appendix.

3. Weak Metrizability

One of the central issues in set-theoretical topology is the question under which conditions a given topological space is *metrizable*. A topological space is metrizable if there is a metric distance function $d : X \times X \to \mathbb{R}_0^+$ such that $\mathcal{B}(x)$, defined in equ.(2), is a neighborhood basis of the prescribed topology.

A natural way to generalize the construction of a topological structure from a metric is to replace the requirement that *all* ε-balls are neighborhoods by the conditions that the neighborhoods are generated by *some* ε-balls. This suggests the following

Definition 1. *A pretopological space* $(X, \mathcal{N})$ *is* weakly metrizable *if there is a metric* $d : X \times X \to \mathbb{R}_0^+$ *and sets* $A, A' \subseteq \mathbb{R}^+$ *such that*

$$\mathcal{B}(x) = \{B_\varepsilon(x)|\varepsilon \in A\} \cup \{B'_\varepsilon(x)|\varepsilon \in A'\} \tag{3}$$

is a neighborhood basis of $(X, \mathcal{N})$.

This definition looks a bit complicated since we explicitly take both types of balls from equ.(1) into account. Thus we have two independent index sets A and A' in equ.(3). The reason for this is that, in constrast to classical topology, where the "open ε-balls" $B_\varepsilon(x)$ are fundamental, there is no *a priori* reason to favor one type of balls over the other in our setting.

The main result about weak metrizability — and probably the reason why this generalization of metrizability to pretopological spaces has received little attention so far — is the following theorem showing that weak metrizability is a quite restrictive condition on the neighborhood filters.

Theorem 1. *Let* (X, d) *be a weakly metric pretopological space. Then it has a neighborhood basis of one of the following types:*

$\{\{x\}\}$;
$\{B_\alpha(x)\}$, *for some constant* $\alpha > 0$;
$\{B'_\alpha(x)\}$, *for some constant* $\alpha > 0$;
$\{B_{\alpha+1/n}(x)|n \in \mathbb{N}\}$, *for some constant* $\alpha \geq 0$.

As an immediate consequence, a weakly metrizable pretopological space is *first countable*, i.e., every point $x \in X$ has a countable neighborhood basis.

We observe that, in general, a space with neighborhood basis of the form (3) does not satisfy the topology axiom (P4). Furthermore, it is well known that metric (pre)topological spaces are topological; the simple argument is given in Appendix B. It follows that a weakly metrizable space must necessarily be topological in order to be metrizable. It is tempting to conjecture that the converse it true as well. However, (P4) is not sufficient: Let X be an arbitrary set and let d be the trivial metric, i.e., $d(x, y) = 1$ whenever $x \neq y$. The neighborhood basis $\{B'_1(x)\}$ generates the indiscrete topology on X since $B'_1(x) = X$ for all X. Thus an indiscrete topological space is weakly metrizable but not metrizable if it contains more than a single point.

Definition 2. *A pretopological space $(X, \mathcal{N})$ in which each point has a smallest neighborhood is called an* Alexandroff space*. The smallest neighborhood of a point $x \in X$ is the* vicinity

$$N(x) = \cap\mathcal{N}(x) = \bigcap_{N\in\mathcal{N}(x)} N \tag{4}$$

Topological Alexandroff spaces were introduced in [1] and are studied in some detail in [3, 31].

Clearly, if $d(\,.\,,\,.\,)$ is a metric then $\beta d(\,.\,,\,.\,)$ is also a metric for all $\beta > 0$. Thus a finite pretopological space is weakly metrizable if and only if there is a metric d on X such that $N(x) = B'_1(x)$ for all $x \in X$. Since there is a finite minimal distance between any two points in this case one can rescale the distance also in such a way that $N(x) = B_1(x)$ for all $x \in X$. Clearly, this $N(x)$ can be interpreted as the vertices adjacent to x. For infinite weakly metrizable Alexandroff spaces we have either $N(x) = B'_1(x)$ or $N(x) = B_1(x)$.

4. Symmetry and Regularity

Metric spaces satisfy stringent regularity and separation conditions. In this and the following section we briefly outline some of the corresponding properties in pretopological spaces and show that weakly metrizable spaces need not satisfy such restrictive conditions. The closure $\overline{A}$ of a set $A \subseteq X$ in the pretopological space $(X, \mathcal{N})$ is defined in appendix A. We remark that a pretopological space is topological if and only if the closure is idempotent, i.e., if $\overline{\overline{A}} = \overline{A}$ for all $A \subseteq X$.

Definition 3. *A pretopological space is*

(R0) weakly regular *if $x \in \overline{\{y\}}$ implies $y \in \overline{\{x\}}$ for all $x, y \in X$ [23];*
(S') weakly symmetric *if $x \in N_y$ for all $N_y \in \mathcal{N}(y)$ implies $y \in N_x$ for all $N_x \in \mathcal{N}(x)$;*
(S) (strongly) symmetric *if $x \in N_y$ for all $N_y \in \mathcal{N}(y)$ implies $\mathcal{N}(x) = \mathcal{N}(y)$;*
(Re) reciprocal *if the fact that $N_x \cap N_y \neq \emptyset$ for all $N_x \in \mathcal{N}(x)$ and all $N_y \in \mathcal{N}(y)$ implies $\mathcal{N}(x) = \mathcal{N}(y)$;*
(R) regular *if every neighborhood $N \in \mathcal{N}(x)$ contains the closure $\overline{N'}$ of a neighborhood $N' \in \mathcal{N}(x)$.*
(tR) t-regular *if every neighborhood $N \in \mathcal{N}(x)$ contains a closed neighborhood $N' = \overline{N'} \in \mathcal{N}(x)$.*
(CR) completely regular *if for every neighborhood $N \in \mathcal{N}(x)$ there is a neighborhood $N' \in \mathcal{N}(x)$ such that N' is completely within[2] N.*

Eduard Čech proved that a pretopological space is semi-uniformizable if and only if it satisfies (S') [7, Thm.23.B.3]. In [15] it is shown that (S) is equivalent to "weak uniformizability". Reciprocal spaces were considered in [14], where (Re) was termed

[2]See Appendix A.

"axiom P"; the regularity axiom (R) was introduced by Fischer [9]. The following result summarizes the relationships between these regularity properties.

Theorem 2. *Let* $(X, \mathcal{N})$ *be a pretopological space. Then*
(Re) $\Longrightarrow$(S) $\Longrightarrow$(S') $\Longleftrightarrow$ (R0) $\Longleftarrow$ (R) $\Longleftarrow$ (tR) $\Longleftarrow$ (CR).
A completely regular pretopological space is topological.

In topological spaces (S') and (S) are equivalent, see e.g. [22]. Furthermore, (R) and (tR) are equivalent in topological spaces since the closure of a set is closed in this case.

It is well known that metrizable spaces are completely regular (in fact, metrizable spaces satisfy the even stronger condition of being "completely normal", see e.g. [21]), and therefore regular. For weak metrizability only a much weaker result holds:

Theorem 3. *A weakly metrizable pretopological space is* (R0)*.*

Weakly metrizable spaces are in general neither symmetric nor regular.
Example. Consider the set $\mathbb{R}^2$ and define a pretopology on $\mathbb{R}^2$ by the neighborhood basis $\{B'_1(x)\}$, i.e., N is a neigborhood of x iff it contains the filled unit disk centered at x. Then $\hat{x} = (0, 1/2)$, for instance, is contained in every neighborhood of the orgin $\hat{o}$, but of course their neighborhoods are not the same, i.e. (S) is not satisfied. The closure $\overline{B'_1(\hat{o})}$ of $B'_1(\hat{o})$ consists of all $y \in \mathbb{R}^2$ such that $B'_1(\hat{o}) \cap B'_1(y) \neq \emptyset$, i.e., $d(\hat{o}, y) \leq 2$. Obviously, the closures $\overline{B'_1(x)}$ do not form a neighborhood basis, and hence this space is not regular.

5. Separation Properties

Metric spaces satisfy not only strong regularity properties but also stringent separation properties: A metrizable space is Hausdorff, see e.g. [21]. Below we introduce the most important separation properties in pretopological spaces and we briefly discuss their mutual relationships:

Definition 4. *A pretopological space* $(X, \mathcal{N})$ *is*

- (T0) *if for all* $x \neq y \in X$ *there is a neighborhood* $N \in \mathcal{N}(x)$ *such that* $y \notin N$ *or a neighborhood* $N' \in \mathcal{N}(y)$ *such that* $x \notin N'$*;*
- (T1) *if for all* $x \neq y \in X$ *there is a neighborhood* $N \in \mathcal{N}(x)$ *such that* $y \notin N$*;*
- (T2) *if for all* $x \neq y \in X$ *there are disjoint neighborhoods* $N_x \in \mathcal{N}(x)$ *and* $N_y \in \mathcal{N}(y)$*;*
- (T2$\frac{1}{2}$) *if for all* $x \neq y \in X$ *there are neighborhoods* $N_x \in \mathcal{N}(x)$ *and* $N_y \in \mathcal{N}(y)$ *such that* $\overline{N_x} \cap \overline{N_y} = \emptyset$*. (*Urysohn*-property).*
- (T3) *if it is* (T0) *and regular.*
- (T3$\frac{1}{2}$) *if it is* (T0) *and completely regular.*

We first establish a few alternative characterizations of the separation properties (T1) and (T2) that highlight the relationships between separation and regularity properties.

Theorem 4. *A pretopological space* $(X, \mathcal{N})$ *is:*

(T1) *if and only if* $\overline{\{x\}} = \{x\}$ *for all* $x \in X$*, i.e., iff "every point is closed";*
(T1) *if and only if it is* (R0) *and* (T0)*;*
(T2) *if and only if every filter converges to at most one point. This is the* Hausdorff*-property.*
(T2) *if and only if it is* (Re) *and* (T0)*.*

The separation axioms introduced above have the same sequence of implications in pretopological spaces as in the more familiar case of topological spaces, see e.g. [21]:

Theorem 5. *For a pretopological space we have the implications*
$(T3\frac{1}{2}) \Longrightarrow (T3) \Longrightarrow (T2\frac{1}{2}) \Longrightarrow (T2) \Longrightarrow (T1) \Longrightarrow (T0)$.

It is well known that none of these implications can be reversed even when restricted to topological spaces. For counterexamples see e.g. [30]. In order to make this contribution self-contained the rather simple proofs are given in Appendix C.

The following result shows that there is a simple relationship between metrizable and weakly metrizable spaces. The real mathematical challenge of course remains to find a characterization of weakly metrizable pretopologies, perhaps along the lines of the celebrated *Nagata-Smirnov-Bing* metrization theorem [5, 19, 26] for topological spaces.

Theorem 6. *A weakly metrizable pretopological space is metrizable if and only if it is* (T1)*.*

6. Recombination Sets

Given two chromosomes x and y the recombination set $\mathcal{R}(x, y)$ consists of all chromosomes that can be obtained by recombining x and y using a given family of crossover operators. Consider the following properties:

(X1) $\{x, y\} \subseteq \mathcal{R}(x, y)$;
(X2) $\mathcal{R}(x, y) = \mathcal{R}(y, x)$;
(X3) For all $z \in \mathcal{R}(x, y)$ holds $|\mathcal{R}(x, z)| \le |\mathcal{R}(x, y)|$.
(X4) $\mathcal{R}(x, x) = \{x\}$;

A generalized recombination structure satisfies (X1) and (X2). The proper recombination structures of homologous crossover satisfy also (X3) and (X4) [12].

It seems natural to interpret $\mathcal{R}(x, y)$ as neighborhoods of x for each $y \in X$. By (X1) we have $x \in \mathcal{R}(x, y)$ for all x, y. Thus the recombination sets form a neighborhood basis if and only if for all x, y, z there is a v such that

$$\mathcal{R}(x, v) \subseteq \mathcal{R}(x, y) \cap \mathcal{R}(x, z) \tag{5}$$

In general this condition will not be satisfied, although a large class of recombination models have this property as we shall see below. We may, however, consider the recombination sets as a *sub-basis* of the neighborhood filters and construct the coarsest pretopology in which the recombination sets are neighborhoods by adding the intersections of any finite number of recombination sets to the basis.

In the case of finite genome sets X we know that there is always a smallest neighborhood $N(x)$, i.e., a minimal element of the neighborhood basis. This is true in general if the neighborhood filters have a finite basis, i.e., in Alexandroff spaces. Provided X is finite we can extract the vicinities directly from the (sub)basis of recombination sets:

$$N(x) = \bigcap_{y \in X} \mathcal{R}(x, y) \tag{6}$$

If X is infinite, however, $N(x)$ defined in equ.(6) need not be a neighborhood of x in general. The intersection of a finite number of neighborhoods of course is again a neighborhood. Equ.(6) however defines neighborhoods if the size of the recombination sets $\mathcal{R}(x, y)$ is bounded.

7. Unequal Crossover

7.1. Unrestricted Unequal Crossover. Let us consider two chromosomes each with a cluster of gene copies. One chromosome with x gene copies and the other with y copies. We assume an extreme form of unequal crossover, namely that a crossover may happen with equal probability at all possible intergenic regions as well as at both ends of the gene cluster. Each possible crossover event produces two recombinant chromosomes. In most cases the recombination event will yield chromosomes with different numbers of gene copies than the original ones. Let $\mathcal{R}_U(x, y)$ be the recombination set, i.e., the set of all possible recombinants between chromosomes with x and y copies of the gene.

By abuse of notation we use x as the symbol for a chromosome with a certain number of gene copies as well as for the number of gene copies on the chromosome. We can employ this simple model because we are dealing with a paralogous cluster. In the more general case one cannot simply ignore the identity and therefore the ordering of the genes.

It is easy to see that the recombination set for this operator is [25]

$$\mathcal{R}_U(x, y) = \{0, \dots, x + y\}\,. \tag{7}$$

We see immediately $\mathcal{R}_U(x, y) \subseteq \mathcal{R}_U(x, z)$ if and only if $y \leq z$. It follows that the recombination sets form a neighborhood basis: simply choose $v \leq \min\{y, z\}$ in equ.(5). Even though X is infinite in this example we see that there is a finite neighborhood basis since the sets $\mathcal{R}_U(x, z)$ for $z > y$ contain $\mathcal{R}_U(x, y)$ and can therefore be omitted from the basis. The pretopology $\mathcal{N}_U$ is therefore Alexandroff, generated by the vicinities

$$N_U(x) = \mathcal{R}_U(x, 0) = \{0, \dots, x\}\,. \tag{8}$$

Next we observe that $N_U(y) \subseteq N_U(x)$ if and only if $y \le x$. This can be rewritten as

$$N_U(y) \subseteq N_U(x) \quad \text{whenever} \quad y \in N_U(x) \tag{9}$$

and hence the recombination space of this model is in fact a *topology*, because every neighborhood of x contains a neighborhood of each of its points (this is equivalent to axiom (P4) in Appendix A).

Another way to see that this model of unequal crossover generates a topological configuration space is to consider the canonical closure operator associated with the recombination operator. From the definition in Appendix A and equ.(8) we find

$$\overline{A} = \{y|A \cap N \neq \emptyset \, \forall N \in \mathcal{N}(y)\} = \{y|A \cap N_U(y) \neq \emptyset\} = \{y \ge \min A\} \tag{10}$$

Obviously, this closure operator is idempotent since $\min \overline{A} = \min A$.

Let us now turn to the separation properties. If $x < y$ then $y \notin N(x)$ by equ.(8), and if $x > y$ then $x \notin N(y)$. Consequently $(X, \mathcal{N}_U)$ is a (T0) space. However, since $N_U(x) \neq \{x\}$ for all $x > 0$ we immediately see that it is not (T1): for instance, there is no neighborhood of 2 that does not contain 1. Theorem 4 implies that (R0) does not hold since (T1) is equivalent to (T0) and (R0). Finally, we use Theorem 3 to conclude that $(X, \mathcal{N}_U)$ is not weakly metrizable.

It is interesting to note that $(X, \mathcal{N}_U)$ is hierarchical in the sense of [8], i.e., $N_U(x) \cap N_U(y)$ equals either $N_U(x)$ or $N_U(y)$ (or $\emptyset$, which does not occur in this case). Hence it is even a *normal* topological space. Nevertheless, it fails to be weakly metrizable. An extension of the notion of normality to pretopological spaces is considered in [20].

The main argument against the existence of a metric in the space of this rather extreme form of unequal crossover is that any two recombination sets and hence neighborhoods share at least $\{0\}$ if not a much larger subset. This is of course a consequence of the assumption that any two chromosomes can have mismatched crossover of any number of gene positions and thus can yield a chromosome without any gene copy, $x = 0$. Hence it is important to ask whether the non-existence of a metric also holds for less extreme forms of unequal crossover.

It can be argued that, since unequal crossover events are rare, small mismatches (by a single unit) are more likely than the multiple mismatches that our equi-probable model allows. Thus, if we suggest that the probability of a single mismatch is p, then the probability of a mismatch by k units might scale as $\sim p^k$ (perhaps chosen from a Poisson distribution). Strictly speaking, insofar as the edges in the configuration space are specified by all nonzero transition probabilities, this crossover model (where edges are weighted by transition probability) is still topologically identical to the Shpak and Wagner model.

7.2. Restricted Unequal Crossover. If we approximate the unequal crossover model by assuming that only the order p transitions, i.e., the single unit mismatches, are allowed we obtain the unequal crossover model of Kruger and Vogel [17]. Here the number of gene-copies changes by at most one compared to the parental chromosome numbers. The recombination sets for this model are

$$\begin{aligned} &\mathcal{R}_{\pm}(x,y) = \{x-1,x,x+1\} \cup \{y-1,y,y+1\} \quad \text{if } x,y \geq 1 \\ &\mathcal{R}_{\pm}(x,0) = \{x-1,x,0,1\} \text{ if } x \geq 1 \\ &\mathcal{R}_{\pm}(0,0) = \{0\} \end{aligned} \tag{11}$$

In this case the recombination sets form only a subbasis, since

$$\mathcal{R}_{\pm}(x,y) \cap \mathcal{R}_{\pm}(x,0) = \{x-1,x\} \tag{12}$$

for $y \gg x \geq 1$. The sets $\{x-1,x\}$, however, do not contain recombination sets in general. Since the cardinality of the recombination sets is finite (it varies between one for $\mathcal{R}_{\pm}(0,0)$ and six if $|x-y| \geq 3$ and $x,y \geq 1$) we know that the vicinities as defined in (6) are neighborhoods. We have explicitly,

$$N_{\pm}(x) = \{x-1,x\} \quad \text{if } x \geq 1 \text{ and } N_{\pm}(0) = \{0\} \tag{13}$$

Again, we see immediately that the pretopology satisfies (T0) since $y \notin N_{\pm}(x)$ whenever $y > x$. On the other hand, (T1) is violated since every neighborhood of 1 contains 0. As in the unrestricted model we conclude that the Kruger-Vogel recombination space is not (R0) and therefore not weakly metrizable.

7.3. Unequal Sister Chromatide Exchange. Axelrod *et al.* [4] consider a model of gene amplification by unequal sister chromatide exchange. The model is based on the following assumptions:

(i) A cell contains $k > 0$ repeats on each sister chromatid. The state space is thus $\mathbb{N} = \{1,2,\dots\}$. Furthermore we have $\mathcal{R}(x,y) = \emptyset$ whenever $x \neq y$.
(ii) Recombinants with $k-s$, $s = -(k-1),\dots,k-1$, occur with frequencies $\sim p^{|s|}$. This gives rise to two variants: $\mathcal{R}_A$ arises by taking all transitions into account, while $\mathcal{R}_a$ is obtained by retaining only the non-recombination case and the $\mathcal{O}(p)$ transitions, i.e., $s = -1, 0, +1$.

Hence we have the recombination sets

$$\begin{aligned} \mathcal{R}_A(x,x) = \{1,2,\dots,2x-1\} \qquad &\mathcal{R}_a(x,x) = \{x-1,x,x+1\} \qquad x \geq 2 \\ &\mathcal{R}_a(1,1) = \{1,1\} \end{aligned} \tag{14}$$

The analysis of this model is very simple because $N(x) = \mathcal{R}(x,x)$ in both variants. Both pretopologies are neither (R0) nor (T0), and hence not weakly metrizable, as the following counterexamples show:

$$\begin{aligned} &N_A(1) = \{1\} &&N_a(1) = \{1\} \\ &N_A(2) = \{1,2,3\} &&N_a(2) = \{1,2,3\} \\ &N_A(3) = \{1,2,3,4,5\} &&N_a(3) = \{2,3,4\} \end{aligned}$$

In both cases $2 \in N(3)$ and $3 \in N(2)$, i.e., the space is not (T0). Furthermore we have $1 \in N(2)$ but $2 \notin N(1)$, contradicting (R0).

7.4. Extended Shpak-Wagner Model. If the genes in question are not undistinguishable copies we obtain a generalization of the unrestricted model 7.1 that takes into account the identity of the genes, and hence also their ordering along the chromosome. This generalization was suggested by an anonymous referee of an early version of this contribution. We represent a genome by a string $\mathbf{x}$ with length ℓ_x such that the letter x_i denotes the i-th gene in $\mathbf{x}$. The size of the alphabet from which $\mathbf{x}$ is built is the number of distinct genes.

The following notation for the initial and terminal substrings of $\mathbf{x}$ will be convenient:

$$\mathbf{x}_i = (x_1, x_2, \dots, x_i) \qquad \text{and} \qquad \mathbf{x}^j = (x_{j+1}, x_{j+2}, \dots, x_{\ell_x}) \tag{15}$$

Note that $\mathbf{x}_0 = \mathbf{x}^{\ell_x} = \emptyset$, $\mathbf{x}_{\ell_x} = \mathbf{x}^0 = \mathbf{x}$ and $(\mathbf{x}_l)^k = (x_{k+1}, \dots, x_l)$, $k \le l$. With this notation it is straight forward to write down the recombination sets for unequal 1-point crossover:

$$\mathcal{R}(\mathbf{x},\mathbf{y}) = \left\{\mathbf{x}_k\mathbf{y}^l \middle| 0 \le k \le \ell_x,\ 0 < l < \ell_y\right\} \cup \left\{\mathbf{y}_l\mathbf{x}^k \middle| 0 \le k \le \ell_x,\ 0 < l < \ell_y\right\} \tag{16}$$

For recombination with the "empty" genome $\emptyset$ which does not contain a member of the gene-cluster in question we have therefore

$$\mathcal{R}(\mathbf{x},\emptyset) = \left\{\mathbf{x}_k \middle| 0 \le k \le \ell_x\right\} \cup \left\{\mathbf{x}^k \middle| 0 \le k \le \ell_x\right\} \tag{17}$$

Setting $l = l_y$ and $l = 0$ in the first and second part of equ.(16), respectively, we see immediately that $\mathcal{R}(\mathbf{x},\emptyset) \subseteq \mathcal{R}(\mathbf{x},\mathbf{y})$ for all $\mathbf{x}$ and $\mathbf{y}$. Thus the vicinity of $\mathbf{x}$ is $N(\mathbf{x}) = \mathcal{R}(\mathbf{x},\emptyset)$. From equ.(17) we obtain immediately

$$N(\mathbf{x}_k) = \{\mathbf{x}_j | 0 \le j \le k\} \cup \left\{(\mathbf{x}_k)^j | 0 \le j \le k\right\} \tag{18}$$

Clearly, substrings of the form $(\mathbf{x}_k)^j$ are not part of $N(\mathbf{x})$ in general, i.e., $N(\mathbf{x}_k) \not\subseteq N(\mathbf{x})$. Thus the extended Shpak-Wagner model is not topological. However, it reduces to the unrestricted unequal crossover model 7.1 if the alphabet consists only of a single letter. In this case we have $(\mathbf{x}_k)^j = \mathbf{x}_{k-j} \in N(\mathbf{x})$.

Consider $\mathbf{y} \in N(\mathbf{x})$ such that $\mathbf{y} \ne \mathbf{x} = \mathbf{x}^0 = \mathbf{x}_{\ell_x}$. Then $\mathbf{y}$ is a strict substring of $\mathbf{x}$, i.e., $\ell_y < \ell_x$. Thus for each $\mathbf{y} \in N(\mathbf{x})$ with $\mathbf{y} \ne \mathbf{x}$ we have $\mathbf{x} \notin N(\mathbf{y})$ because the lengths of the strings must satisfy $\ell_z \le \ell_y < \ell_x$ for all $\mathbf{z} \in N(\mathbf{y})$. Thus the model is (T0) but not (T1). Weak metrizability therefore fails again because of the lack of symmetry.

8. Conclusions

To our knowledge this is a first rigorous proof about the non-metric nature of configuration spaces induced by unequal crossover operators as suggested in Shpak and Wagner [25]. This is of particular interest since the recombination structure has the strict symmetry property (X2) which, however, does not translate into sufficient symmetry of the corresponding pretopology. In contrast, point mutation space and the spaces of homologous recombination are weakly metrizable [12, 28, 29].

Another example of a non-symmetric configuration space arises in the context of RNA evolution: Wolfgang Schnabl [24] considered a model of RNA replication with insertions and deletions in which insertions are restricted to duplications of substrings, while arbitrary subsequences can be deleted.

While asymmetric exchange rates and transmission bias can be generated by unequal edge weights on a metric configuration space, there need not be any such bias. In contrast, non-metric configuration spaces necessarily induce a bias in transmission dynamics, because fundamental symmetries between any two points do not hold. We argue that in some sense configuration space non-metricity and non-topology are more fundamental asymmetries than those induced by unequal edge weights on a metric topological space.

The results on recombination spaces reported in the previous section suggest that a similar principle of non-metricity may hold for the recombination operator of genetic programming [16]. This may be an explanation for the tendency of genetic programming operators to "bloat" the codes they are evolving, i.e., accumulate non-functional code.

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Appendix A: Pretopological Spaces and Closure Operators

A pretopological space is conveniently defined by specifying (a basis of) its neighborhood structure $\mathcal{N}(x)$.

(P1) $x \in N$ for all $N \in \mathcal{N}(x)$.
(P2) If $N, N' \in \mathcal{N}(x)$ then there is $N'' \in \mathcal{N}(x)$ such that $N'' \subseteq N \cap N'$.
(P3) If $N \in \mathcal{N}(x)$ and $N \subseteq N'$, then $N' \in \mathcal{N}(x)$.
(P4) For every $N \in \mathcal{N}(x)$ there is $N' \in \mathcal{N}(x)$ such that for all $y \in N'$ there is a neighborhood $N'' \in \mathcal{N}(y)$ satisfying $N'' \subseteq N$.

A subset $\mathcal{N}(x) \subseteq \mathfrak{P}(X)$ is a *neighborhood basis* of a pretopological space $(X, \mathcal{N})$ if it satisfies (P1) and (P2). If (P3) is satisfied as well, $\mathcal{N}(x)$ is the *neighborhood filter.* A pretopological space whose neighborhood filter or neighborhood basis satisfies (P4) is a topological space [2, Thm.IX'].

A *filter* on X is a collection $\mathcal{F}$ of subsets of X satisfying

(F1) $F \in \mathcal{F}$ implies $F \neq \emptyset$
(F2) If $F, F' \in \mathcal{F}$ then there is $F'' \in \mathcal{F}$ such that $F'' \subseteq F \cap F'$.
(F3) If $F \in \mathcal{F}$ and $F \subseteq F'$, then $F' \in \mathcal{F}$.

If only (F1) and (F2) are satisfied we speak of a *filter base.*

A filter $\mathcal{F}$ is said to *converge* to x in $(X, \mathcal{N})$ if it is finer than the neighborhood filter of x, in symbols if $\mathcal{N}(x) \subseteq \mathcal{F}$. One writes $\mathcal{F} \to x$ in this case.

In a pretopological space $(X, \mathcal{N})$ there is a natural *closure operator* defined by

$$\overline{A} = \{y \in X \big| A \cap N \neq \emptyset \text{ for all } N \in \mathcal{N}(y)\} \tag{19}$$

Conversely, a closure operator satisfying the axioms

(K0) $\overline{\emptyset} = \emptyset$.
(K1) $A \subseteq \overline{A}$.
(K2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

for all $A, B \in \mathfrak{P}(X)$ defines a unique pretopological space [13]. The corresponding neighborhood filters are determined by

$$\mathcal{N}(x) = \{A \subseteq X | x \notin \overline{X \setminus A}\}. \tag{20}$$

This pretopology is topological if and only if the closure is idempotent, i.e., iff and only if

(K3) $\overline{\overline{A}} = \overline{A}$.

A set A is said to be *completely within* B in a pretopological space $(X, \mathcal{N})$ if there is a continuous function $\varphi : (X, \mathcal{N}) \to [0, 1]$ (the real unit interval with the usual topology) such that $\varphi(A) \subseteq \{0\}$ and $\varphi(X \setminus B) \subseteq \{1\}$. Obviously, if A is completely within B we have $A \subseteq B$, the empty set $\emptyset$ is completely within every non-empty set, and every set is completely within the entire space X.

Appendix B: Metric Distance Measures

Let X be an arbitrary set. A function $d : X \times X \to \mathbb{R}_0^+ \cup \{\infty\}$ that satisfies

(M0) $d(x, x) = 0$ for all $x \in V$.
(M1) $d(x, y) = 0$ implies $x = y$.
(M2) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in V$.
(M3) $d(x, y) = d(y, x)$ for all $x, y \in V$.

is called a *metric* on X and (X, d) is called a *metric space*.

Lemma 1. *The neighborhood bases $\mathcal{B}(x) = \{B_\varepsilon(x) | \varepsilon > 0\}$ define a topology.*

Proof. Consider an arbitrary ε-ball $B_\varepsilon(x)$ and $y \in B_\varepsilon(x)$. Then $d(x, y) = \eta < \varepsilon$ by (M2). Thus there is a ζ such that $0 < \zeta < \varepsilon - \eta$. Now consider a point $z \in B_\zeta(y)$. We have $d(x, z) \leq d(x, y) + d(y, z) < \eta + \zeta < \varepsilon$ and hence $B_\zeta(y) \subseteq B_\varepsilon(x)$. Thus the "topology axiom" (P4) is satisfied. □

Appendix C: Proofs

Theorem 1. *Let (X, d) be a weakly metric pretopological space. Then it has a neighborhood basis of one of the following types:*

$\{\{x\}\}$;
$\{B_\alpha(x)\}$, *for some constant* $\alpha > 0$;
$\{B'_\alpha(x)\}$, *for some constant* $\alpha > 0$;
$\{B_{\alpha+1/n}(x) | n \in \mathbb{N}\}$, *for some constant* $\alpha \geq 0$.

Proof. Consider the basis in equ.(3) and define $\alpha = \inf A$ and $\alpha' = \inf A'$. First suppose $\alpha < \alpha'$. Then for each $\varepsilon' \in A'$ there is an $\varepsilon \in A$ such that $\varepsilon < \varepsilon'$, and hence $B_\varepsilon(x) \subseteq B'_\varepsilon(x)$. It follows that we can omit the filled balls and $\{B_\varepsilon(x)|\varepsilon \in A\}$ is already a basis. If $\alpha \in A$, then $\{B_\alpha(x)\}$ is also basis since $B_\alpha(x)$ is a basis element and it is contained in all $B_\varepsilon(x)$, $\varepsilon \in A$. If $\alpha \notin A$ then there is no finite basis. We can then find $n \in \mathbb{N}$ for every $\varepsilon \in A$ such that $\varepsilon > \alpha + 1/n$ and $\varepsilon' \in A$ such that $\varepsilon' < \alpha + 1/n$. Thus $B_{\alpha+1/n}(x)$ is an element of the neighborhood filter for all $n \in \mathbb{N}$ and each $B_\varepsilon(x)$ contains an element of of $\{B_{\alpha+1/n}(x)|n \in \mathbb{N}\}$, which is therefore itself a neighborhood basis. An analogous argument can be made for $\alpha' < \alpha$. If $\alpha' \in A'$ then $\{B'_{\alpha'}(x)\}$ is a basis, otherwise $\{B'_\varepsilon(x)|\varepsilon \in A'\}$ is a basis. In this case, for each $\varepsilon \in A'$ there is an $n \in \mathbb{N}$ such that $\alpha' + 1/n < \varepsilon$ and $\varepsilon' \in A'$ such that $\varepsilon < \alpha' + 1/n$. We have $B'_{\varepsilon'}(x) \subset B_{\alpha'+1/n}(x) \subset B'_{\alpha'+1/n}(x) \subset B'_\varepsilon(x)$, i.e., $\{B_{\alpha'+1/n}(x)|n \in \mathbb{N}\}$ is also a neighborhood basis of x.
It remains to deal with the case $\alpha = \alpha'$. If $\alpha \in A$ then $\{B_\alpha(x)\}$ is a basis since $B_\alpha(x) \subset B'_\alpha(x) \subseteq B'_\varepsilon(x)$ for all $\varepsilon > \alpha$. If $\alpha \notin A$ and $\alpha \in A'$ then $\{B'_\alpha(x)\}$ is a basis since $B'_\alpha(x) \subset B_\varepsilon(x)$ for all $\varepsilon > \alpha$. Finally, if α is contained neither in A nor in A' we can use the same argument as in the previous paragraph to show that both $\{B'_{\alpha+1/n}(x)|n \in \mathbb{N}\}$ and $\{B_{\alpha+1/n}(x)|n \in \mathbb{N}\}$ are neighborhood bases. □

Reciprocity can be expressed in terms of filter-convergence: A pretopological space is (Re) provided that $\mathcal{N}(x) = \mathcal{N}(y)$ whenever there exists a filter $\mathcal{F}$ that converges to both x and y.

The following property of filter convergence is a useful tool in subsequent proofs.

Lemma 1. *Suppose $\mathcal{F} \to x$, i.e., $\mathcal{N}(x) \subseteq \mathcal{F}$. Then $x \in \overline{F}$ for all $F \in \mathcal{F}$.*

Proof. Since $\mathcal{F}$ is finer than $\mathcal{N}(x)$ we have $N_x \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ and all $N_x \in \mathcal{N}(x)$. Recall that by definition $z \in \overline{F}$ iff $F \cap N_z \neq \emptyset$ for all $N_z \in \mathcal{N}(z)$. It follows that $x \in \overline{F}$ for all $F \in \mathcal{F}$. □

Theorem 2. *Let $(X,\mathcal{N})$ be a pretopological space. Then*
(Re) $\Longrightarrow$(S) $\Longrightarrow$(S') $\Longleftrightarrow$ (R0) $\Longleftarrow$ (R) $\Longleftarrow$ (CR).
A completely regular pretopological space it topological.

Proof. (R0)$\Longleftrightarrow$(S') We have $y \in \overline{\{x\}}$ iff $x \in N_y$ for all $N_y \in \mathcal{N}(y)$. If (R0) holds, we have $x \in \overline{\{y\}}$ which is true if and only if $y \in N_x$ for all $N_x \in \mathcal{N}(y)$, i.e., (S') holds. The converse follows analogously.
(Re)$\Longrightarrow$(S) If $y \in N_x$ for all $N_x \in \mathcal{N}(x)$ then $N_x \cap N_y \neq \emptyset$ and all $N_y \in \mathcal{N}(y)$ and (Re) implies $\mathcal{N}(x) = \mathcal{N}(y)$, i.e., (S) holds.
(R)$\Longrightarrow$(R0) Suppose $y \in N_x$ for all $N_x \in \mathcal{N}(x)$. In other words, the discrete filter $\dot{y}$ is finer than $\mathcal{N}(x)$. Lemma 1 therefore implies $x \in \overline{F}$ for all $F \in \dot{y}$. Since $\dot{y}$ is by definition finer that $\mathcal{N}(y)$ we conclude that for each $N_y \in \mathcal{N}(y)$ there is $F \in \dot{y}$ such that $F \subseteq N_y$ and hence $x \in \overline{F}$ implies $x \in \overline{N_y}$. Regularity means that the $\overline{N_y}$ form a basis of the neighborhood filter $\mathcal{N}(y)$ and hence $x \in N_y$ for every neighborhood N_y.

Consequently, the space satisfies (S') and, equivalently, (R0).
(tR)$\Longrightarrow$(R) follows directly from the definition.
(CR)$\Longrightarrow$(R) It suffices to show that "A completely within B" implies $\overline{A} \subseteq B$. This is obvious for $A = \emptyset$ or $B = X$. Hence we may assume that $A \neq \emptyset$ and $B \neq X$. In this case we have $\varphi(A) = \{0\}$, $\varphi(X \setminus B) = \{1\}$, and for every $\epsilon > 0$ and every $x \in X$ there is $N \in \mathcal{N}(x)$ such that $\varphi(N) \subseteq I_\epsilon(\varphi(x)) := (\varphi(x) - \epsilon, \varphi(x) + \epsilon) \cap [0, 1]$ since φ is continuous. Now suppose $x \in \overline{A}$, i.e., $N \cap A \neq \emptyset$ for all $N \in \mathcal{N}(x)$. Hence $\varphi(A) = \{0\}$ implies $0 \in \varphi(N) \subseteq I_\epsilon(\varphi(x))$ and consequently $0 \leq \varphi(x) < \epsilon$. In particular $\varphi(x) \neq 1$, which implies $x \notin (X \setminus B)$ and hence $x \in B$.
Let $(X, \mathcal{N})$ be completely regular. For each $N \in \mathcal{N}(x)$, $N \neq X$, there is $N' \in \mathcal{N}(x)$ such that $N' \ll N$, i.e., $N' \subset N$, and there is a continuous function $\varphi : X \to [0, 1]$ such that $\varphi(N') = \{0\}$ and $\varphi(X \setminus N) = \{1\}$. Now consider $z \in N'$. Continuity of φ implies that for all $\epsilon > 0$ there is $N_z \in \mathcal{N}(z)$ such that $\varphi(N_z) \subseteq I_\epsilon(\varphi(z)) = [0, \epsilon]$; hence $N_z \cap (X \setminus N) = \emptyset$, i.e., $N_z \subseteq N$. Thus N contains a neighborhood of every point in N_z; this is axiom (P4).
(CR)$\Longrightarrow$(tR) We use that (CR) implies that the space is topological and hence the closure of every set is closed. Therefore (R) and (tR) are equivalent. □

Theorem 3. *A weakly metrizable pretopological space satisfies* (R0)*.*

Proof. We have $x \in \overline{\{y\}}$ iff $y \in B_\varepsilon(x)$ for all $\varepsilon \in A$ and $y \in B'_{\varepsilon'}(x)$ for all $\varepsilon' \in A'$. Equivalently, $d(x, y) < \varepsilon$ and $d(x, y) \leq \varepsilon'$ for all $\varepsilon \in A$ and $\varepsilon' \in A'$. This is in turn equivalent to $x \in B_\varepsilon(y)$ for all $\varepsilon \in A$ and $x \in B'_{\varepsilon'}(y)$ for all $\varepsilon' \in A'$, i.e., $y \in \overline{\{x\}}$. □

Theorem 4. *A pretopological space $(X, \mathcal{N})$ is:*

(T1) *if and only if $\overline{\{x\}} = \{x\}$ for all $x \in X$, i.e., iff "every point is closed";*
(T1) *if and only if it is* (R0) *and* (T0)*;*
(T2) *if and only if every filter converges to at most one point. This is the* Hausdorff*-property.*
(T2) *if and only if it is* (Re) *and* (T0)*.*

Proof. (i) $y \in \overline{\{x\}}$ iff $x \in N$ for all $N \in \mathcal{N}(y)$. If $y \neq x$ and the space is (T1) then there is $N' \in \mathcal{N}(y)$ such that $x \notin N'$, a contradiction. Conversely, if $\overline{\{x\}} = \{x\}$ then there is no y such that $x \in N$ for all $N \in \mathcal{N}(y)$, i.e., there is $N \in \mathcal{N}(y)$ with $y \notin N'$, i.e. (T1) holds.
(ii) Suppose $(X, \mathcal{N})$ is (T1). Then (T0) and (R0) are trivially satisfied since $\overline{\{x\}} = \{x\}$, i.e., $y \in \overline{\{x\}}$ implies $y = x$. Now suppose the space is (T0) and (R0) and consider $x \neq y \in X$. Without loosing generality we may assume that there is $N' \in \mathcal{N}(x)$ such that $y \notin N'$, and hence $x \notin \overline{\{y\}}$. Now (R0) implies $y \notin \overline{\{x\}}$ and thus there is $N'' \in \mathcal{N}(y)$ with $x \notin N''$; thus (T1) is satisfied.
(iv) There is a filter converging to both x and y if and only if there exists a filter $\mathcal{F}$ that is finer than both $\mathcal{N}(x)$ and $\mathcal{N}(y)$, i.e., if and only if for all $N' \in \mathcal{N}(x)$ and $N'' \in \mathcal{N}(y)$ we have $N', N'' \in \mathcal{F}$ and hence $N' \cap N'' \neq \emptyset$, i.e., if and only if x and y

do not have disjoint neighborhoods.
(v) Suppose $(X,\mathcal{N})$ is (T2). Then it is clearly also (T1). Thus there is no $y \neq x$ such that $y \in N$ for all $N \in \mathcal{N}(x)$ and hence (Re) is trivially satisfied. Conversely, suppose the space is (T0) and (Re) but not Hausdorff. Then there are points $x \neq y$ and a filter $\mathcal{F}$ that converges to both x and y. Thus (Re) implies $\mathcal{N}(x) = \mathcal{N}(y)$ and therefore each $N \in \mathcal{N}(x)$ contains both x and y, contradicting (T0). Consequently a reciprocal (T0)-space is (T2). $\square$

Theorem 5. (T3$\frac{1}{2}$) $\Longrightarrow$(T3) $\Longrightarrow$(T2$\frac{1}{2}$) $\Longrightarrow$(T2) $\Longrightarrow$(T1) $\Longrightarrow$(T0).

Proof. The implications (T2) $\Longrightarrow$(T1) $\Longrightarrow$(T0) follow immediately from the definitions. Since $N \subseteq \overline{N}$ we conclude that (T2$\frac{1}{2}$) implies (T2). Furthermore, (T3$\frac{1}{2}$) implies (T3) because (CR) implies (R), see theorem 2.
In order to show that (T3) implies (T2$\frac{1}{2}$) we procede in two steps: First we show that a (T3) space is (T2). Suppose we have a regular (T0) space and suppose there is a filter $\mathcal{F}$ that converges to two distinct points x and y. Then both x and y are contained in $\overline{F}$ for all $F \in \mathcal{F}$ and thus $x, y \in \overline{N_x}$ for each $N_x \in \mathcal{N}(x)$ and $x, y \in \overline{N_y}$ for each $N_y \in \mathcal{N}(y)$. In a regular space this implies that both x and y are contained in all neighborhoods of both x and y. This contradicts (T0), hence $(X,\mathcal{N})$ must be Hausdorff, and equivalently, (T2).
Any two points of a (T3) thus have neighborboods N_x and N_y that are disjoint. The regularity axiom now guarantees that there are neighborhoods N'_x and N'_y such that $\overline{N'_x} \subseteq N_x$ and $\overline{N'_y} \subseteq N_y$, and hence $\overline{N'_x} \cap \overline{N'_y} = \emptyset$, i.e., (T2$\frac{1}{2}$) is satisfied. $\square$

Theorem 6. *A weakly metrizable pretopological space is metrizable if and only if it is* (T1).

Proof. All metrizable spaces are (T1), hence the condition is necessary. Conversely, suppose $(X,\mathcal{N})$ is weakly metrizable and (T1). For each point $x \in X$ we distiguish two cases: (i) x is isolated, i.e., there is minimum distance δ_x between x and any other point of X. Then (T1) implies $N(x) = B_{\delta_x}(x) = \{x\}$. Clearly, the collection of all ε-balls form a neighborhood basis. (ii) There is a sequence (x_n) of points such that $\delta_n = d(x, x_n) \to 0$. Then $B_{\delta_n}(x)$ is the largest ball around x that does not contain x_n. By (T1) there is a neighborhood N of x that does not contain x_n, and weak metrizability implies that N contains an ε-ball. Therefore $B_{\delta_n}(x)$ must be a neighborhood of x. Now consider $\varepsilon > 0$. There is $n \in \mathbb{N}$ such that $\varepsilon > \delta_n$, hence $B_\varepsilon(x)$ is a neighborhood of x for all $\varepsilon > 0$. It follows that $(X,\mathcal{N})$ is metrizable. $\square$