# Basic Properties of Filter Convergence Spaces

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**Abstract.** This technical report summarized facts from the basic theory of filter convergence spaces and gives detailed proofs for them. Many of the results collected here are well known for various types of spaces. We have made no attempt to find the original proofs.

## 1. Introduction

Mathematical notions such as *convergence*, *continuity*, and *separation* are, at textbook level, usually associated with topological spaces. It is possible, however, to introduce them in a much more abstract way, based on axioms for convergence instead of neighborhood. This approach was explored in seminal work by Choquet [4], Hausdorff [12], Katětov [14], Kent [16], and others. Here we give a brief introduction to this line of reasoning. While the material is well known to specialists it does not seem to be easily accessible to non-topologists. In some cases we include proofs of elementary facts for two reasons: (i) The most basic facts are quoted without proofs in research papers, and (ii) the proofs may serve as examples to see the rather abstract formalism at work.

## 2. Sets and Filters

Let $X$ be a set, $\mathfrak{P}(X)$ its power set, and $\mathcal{H} \subseteq \mathfrak{P}(X)$. The we define

$$\begin{aligned} \mathcal{H}^* &= \{A \subseteq X | (X \setminus A) \notin \mathcal{H}\} \\ \mathcal{H}^\# &= \{A \subseteq X | \forall Q \in \mathcal{H} :\ A \cap Q \neq \emptyset\} \end{aligned} \tag{1}$$

The set systems $\mathcal{H}^*$ and $\mathcal{H}^\#$ are called the *conjugate* and the *grill* of $\mathcal{H}$, respectively. One easily verifies $\mathcal{H}^{**} = \mathcal{H}$.

The set $\mathcal{H} \subseteq \mathfrak{P}(X)$ is *isotone* if $A \in \mathcal{H}$ and $A \subseteq B \subseteq X$ implies $B \in \mathcal{H}$. If $\mathcal{H}$ is isotone, then $\mathcal{H}^* = \mathcal{H}^\#$. $\mathcal{H}^*$ is isotone if and only if $\mathcal{H}$ is isotone.

Let $\mathcal{F}, \mathcal{G} \in \mathfrak{P}(X)$. If $\mathcal{F} \subseteq \mathcal{G}$ we say that $\mathcal{G}$ is *finer* than $\mathcal{F}$ and $\mathcal{F}$ is *coarser* than $\mathcal{G}$. Note that $\mathcal{F} \subseteq \mathcal{G}$ implies $\mathcal{G}^* \subseteq \mathcal{F}^*$.

**Definition 1.** *A* filter basis *on $X$ is a set $\mathcal{F} \subseteq \mathfrak{P}(X)$ satisfying the axioms*

(F1) *$F \in \mathcal{F}$ implies $F \neq \emptyset$,*
(F2) *$F_1, F_2 \in \mathcal{F}$ implies that there exists $F_3 \in \mathcal{F}$ such that $F_3 \subseteq F_1 \cap F_2$*

*If, in addition,*

(F3) *$\mathcal{F}$ is isotone,*

*the set system $\mathcal{F}$ is called a* filter *on $X$. Each filter basis $\mathcal{F}$ uniquely defines a filter which we will denote by $\mathcal{F}\uparrow$. A filter $\mathcal{U}$ is an* ultrafilter *if there is no filter $\mathcal{F} \neq \mathcal{U}$ that is finer than $\mathcal{U}$.*

For a filter $\mathcal{F}$, axiom (F2) can be replaced by

(F2') $F_1, F_2 \in \mathcal{F} \Longrightarrow F_1 \cap F_2 \in \mathcal{F}$.

Each filter basis $\mathcal{F}$ defines a unique filter which we denote by $\mathcal{F}\uparrow$. We denote the set of filters on $X$ by $\Phi X$. The *discrete filter* of $x$ is $\dot{x} = \{A \subseteq X | x \in A\}$. Analogously we write $\dot{F} = \{A \subseteq X | F \subseteq A\}$. Note that $\dot{A} = \{A\}\uparrow$. Hence, we have $\mathcal{F}\uparrow = \bigcup_{F \in \mathcal{F}} \dot{F}$ for any filter basis $\mathcal{F}$.

If $\mathcal{F}$ is a filter, then

$$\mathcal{F}\downarrow = \bigcap \big\{F \in \mathcal{F}\big\} \neq \emptyset \,. \tag{2}$$

Clearly, $\dot{x}\downarrow = \{x\}$. Note that, in general, $\mathcal{F}\downarrow \notin \mathcal{F}$ if $X$ is an infinite set. For instance, the $\varepsilon$-neighborhoods around a point $x \in \mathbb{R}^n$ form a filter basis, the intersection of all neighborhoods of $x$, however, is $\{x\}$ which itself if not a neighborhood of $x$ w.r.t. the standard topology on $\mathbb{R}^n$.

Two filters $\mathcal{F}$ and $\mathcal{G}$ are *disjoint* if there is $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \cap G = \emptyset$. If $\mathcal{F}$ and $\mathcal{G}$ are not disjoint, there is a uniquely defined coarsest filter that is finer than both $\mathcal{F}$ and $\mathcal{G}$:

$$\mathcal{F} \vee \mathcal{G} = \big\{H = F \cap G \big| F \in \mathcal{F},\, G \in \mathcal{G}\big\} \tag{3}$$

If $\mathcal{F}$ and $\mathcal{G}$ are disjoint we write $\mathcal{F} \vee \mathcal{G} = \emptyset$. We have $\mathcal{F} \vee \mathcal{G} \subseteq \mathcal{H}$ if and only if $\mathcal{F} \subseteq \mathcal{H}$ and $\mathcal{G} \subseteq \mathcal{H}$.

## 3. Convergence

### 3.1. Axioms of Filter Convergence.

**Definition 2.** *Let $X$ be a set, $\Phi X$ the set of filters on $X$, and $\mathsf{q} \subseteq \Phi X \times X$ a relation. We will write $\mathcal{F} \to_{\mathsf{q}} x$, "the filter $\mathcal{F}$ converges to $x$", if*

(C0) *$\mathcal{F} \to_{\mathsf{q}} x$ and $\mathcal{F} \subset \mathcal{G}$ implies $\mathcal{G} \to_{\mathsf{q}} x$, i.e., if $\mathcal{F}$ converges to $x$, then every finer filter also converges to $x$.*

The *limes* of $\mathcal{F}$ is the set

$$\lim \mathcal{F} = \{x \in X | \mathcal{F} \to_{\mathsf{q}} x\} \tag{4}$$

By $(\mathsf{C0})$ $\mathcal{F} \subseteq \mathcal{G}$ implies $\lim \mathcal{F} \subseteq \lim \mathcal{G}$. Set $\mathrm{Conv(X)} = \{x \in X | \exists \mathcal{F} : \mathcal{F} \to_{\mathsf{q}} x\}$. It will be useful to extend the notion of convergence to filter bases: We say a filter basis $\mathcal{F} \to_{\mathsf{q}} x$ if and only if the filter $\mathcal{F}\!\uparrow \to_{\mathsf{q}} x$.

**Definition 3.** *Let $X$ be a set, $\Phi X$ the set of filters on $X$, and $\mathsf{q} \subseteq \Phi X \times X$ a relation. The pair $(X, \mathsf{q})$ is a* generalized convergence space *if* $(\mathsf{C0})$ *and*

$(\mathsf{C1})$ $\dot{x} \to_{\mathsf{q}} x$ *for* $x \in X$

*is true.*

Axiom $(\mathsf{C1})$ could be replaced e.g. by $\mathrm{Conv(X)} = X$.

**3.2. Neighborhood.** The notion of "neighborhood" can be introduced by means of the following construction.

**Definition 4.** *Suppose $(X, \mathsf{q})$ satisfies* $(\mathsf{C0})$*. Then*

$$\mathcal{N}_{\mathsf{q}}(x) = \bigcap \{\mathcal{F} \in \Phi X | \mathcal{F} \to_{\mathsf{q}} x\} \tag{5}$$

*is called the* neighborhood filter *of $x \in X$. We call a set $N \in \mathcal{N}_{\mathsf{q}}(x)$ a* neighborhood *of $x$.*

Note that a filter that converges to $x$ is by definition finer than the neighborhood filter of $x$. In other words, for each neighborhood $N \in \mathcal{N}_{\mathsf{q}}(x)$ and each filter $\mathcal{F} \to_{\mathsf{q}} x$, there is a set $F \in \mathcal{F}$ such that $F \subseteq N$.

**3.3. Closure and Interior.** The notions of the *closure* $\mathsf{cl}(A)$ and the *interior* $\mathsf{int}(A)$ of a set $A \subseteq X$ can be defined in terms of convergence. The notions of open and closed sets are related to closure and interior operators in a natural way.

**Definition 5.** *Suppose $(X, \mathsf{q})$ satisfies* $(\mathsf{C0})$*. Then we define*

$$\begin{aligned} \mathsf{cl}(A) &= \left\{x \in X \middle| \exists \mathcal{F} \in \Phi X : A \in \mathcal{F} \text{ and } \mathcal{F} \to_{\mathsf{q}} x\right\} \\ \mathsf{int}(A) &= \left\{x \in A \middle| \mathcal{F} \to_{\mathsf{q}} x \text{ implies } A \in \mathcal{F}\right\} \end{aligned} \tag{6}$$

Even more general definitions of closure operators do not necessarily rely on convergence, see e.g. [10].

**Theorem 1.** *Suppose $(X, \mathsf{q})$ satisfies* $(\mathsf{C0})$*. Then the closure and interior operators satisfy*

- $(\mathsf{0})$ $X \setminus \mathsf{int}(A) = \mathsf{cl}(X \setminus A)$ *and, equivalently,* $X \setminus \mathsf{cl}(A) = \mathsf{int}((X \setminus A))$.
- $(\mathsf{I'})$ $\mathsf{cl}(X) = \mathrm{Conv}(X)$, $\mathsf{int}(X) = X$, $\mathsf{int}(\emptyset) = X \setminus \mathrm{Conv}(X)$, $\mathsf{cl}(\emptyset) = \emptyset$.
- $(\mathsf{II'})$ $A \subseteq B$ *implies* $\mathsf{cl}(A) \subseteq \mathsf{cl}(B)$ *and* $\mathsf{int}(A) \subseteq \mathsf{int}(B)$.

*Axiom* $(\mathsf{C1})$ *is then equivalent to*

- $(\mathsf{III})$ $A \subseteq \mathsf{cl}(A)$ *for all* $A \subseteq X$.

*If* $(\mathsf{C0})$ *and* $(\mathsf{C1})$ *hold, we have in addition*

- $(\mathsf{I})$ $\mathsf{cl}(X) = X$ *and* $\mathsf{int}(\emptyset) = \emptyset$.

(II) $\mathsf{cl}(A) \cup \mathsf{cl}(B) = \mathsf{cl}(A \cup B)$ *and* $\mathsf{int}(A) \cap \mathsf{int}(B) = \mathsf{int}((A \cap B))$.

*Proof.* We first show that the definition of the closure operator is equivalent to

$$x \in \mathsf{cl}(A) \qquad \Longleftrightarrow \qquad \exists \mathcal{F} : \mathcal{F} \to_{\mathsf{q}} x \text{ and } A \in \mathcal{F}^* \tag{7}$$

In order to prove equ.(7) we observe that $A \in \mathcal{F}$ implies $X \setminus A \notin \mathcal{F}$ by (F1) and hence $A \in \mathcal{F}^*$; Conversely, if $\mathcal{F} \to_{\mathsf{q}} x$ and $A \in \mathcal{F}^*$, then $\mathcal{F}^* = \mathcal{F}^\#$ implies that $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Hence there is a filter $\mathcal{G}$ with basis $\{F \cap A | F \in \mathcal{F}\}$, which is finer than $\mathcal{F}$, and hence converges to $x$, and contains $A$.
Equ.(7 implies that $x \notin \mathsf{cl}(A)$ is equivalent to $\mathcal{F} \to_{\mathsf{q}} x \implies A \notin \mathcal{F}^*$. Recalling that $A \notin \mathcal{F}^*$ is equivalent to $X \setminus A \in \mathcal{F}$, we see that $x \notin \mathsf{cl}(A)$ is equivalent to $x \in \mathsf{int}((X \setminus A))$, i.e., $X \setminus \mathsf{cl}(A) = \mathsf{int}(X \setminus A)$, and (0) holds.
Property (I) follows immediately from the definitions, property (II) follows from the isotony of filters.
(C1), $\dot{x} \to_{\mathsf{q}} x$, implies $\{x\} \in \dot{x}$ and hence $A \in \dot{x}$ whenever $x \in A$. Thus $x \in A$ implies $x \in \mathsf{cl}(\{x\})$. Now we use (II): $\{x\} \subseteq A$ implies $x \in \mathsf{cl}(\{x\}) \subseteq \mathsf{cl}(A)$ and hence $A \subseteq \mathsf{cl}(A)$. Conversely, $x \in \mathsf{cl}(\{x\})$ implies that there is filter containing $\{x\}$ that converges to $x$. This filter must contain all sets that contain $x$, i.e., it coincides with $\dot{x}$. Thus $\dot{x} \to_{\mathsf{q}} x$.
If (III) holds, then $X = \mathrm{Conv}(X)$. Thus (I) simplifies to (I').
The inclusion rules (IV) now follow immediately from the definitions of closure and interior and the filter axioms: For instance, if $x \in \mathsf{int}((A \cap B))$ then $\mathcal{F} \to_{\mathsf{q}} x$ implies $A \cap B \in \mathcal{F}$. Since $A \cap B \subseteq A$, isotony of $\mathcal{F}$ implies $A, B \in \mathcal{F}$. Thus $x \in \mathsf{int}(A)$ and $x \in \mathsf{int}(B)$, i.e., $x \in \mathsf{int}(A) \cap \mathsf{int}(B)$. The other three assertions follow by analogous arguments.
If $x \in \mathsf{int}(A) \cap \mathsf{int}(B)$ we have $\mathcal{F} \to_{\mathsf{q}} x$ implies $A \in \mathcal{F}$ and $\mathcal{F} \to_{\mathsf{q}} x$ implies $B \in \mathcal{F}$. In other words, if $x \in A \cap B$ then $\mathcal{F} \to_{\mathsf{q}} x$ implies both $A \in \mathcal{F}$ and $B \in \mathcal{F}$, and hence, by (F2'), $A \cap B \in \mathcal{F}$. Thus $x \in \mathsf{int}((A \cap B))$. Therefore $\mathsf{int}(A) \cap \mathsf{int}(B) \subseteq \mathsf{int}((A \cap B))$. Conversely, we have $A \cap B \subseteq A$ implies $\mathsf{int}((A \cap B)) \subseteq \mathsf{int}(A)$ and $A \cap B \subseteq A$ implies $\mathsf{int}((A \cap B)) \subseteq \mathsf{int}(B)$. Thus $\mathsf{int}((A \cap B)) \subseteq \mathsf{int}(A) \cap \mathsf{int}(B)$. Consequently we have $\mathsf{int}((A \cap B)) = \mathsf{int}(A) \cap \mathsf{int}(B)$. The corresponding result for the closure operator now follows from (0). ☐

Theorem 1 establishes the basic properties of a generalized closure operator. It also strongly suggests that spaces that do not satisfy at least (C0) and (C1) may have counter-intuitive properties such as the non-empty interior of the empty set, or the fact that the closure of a set $A$ can be a non-trivial subset of $A$. The use of (I), (II), and (III) as axioms goes back to Hausdorff's work [12]. We shall see in section B.6 that a closure operator satisfying these axioms uniquely determines a pretopological spaces.

### 3.4. Open and Closed Sets.

**Definition 6.** *Let* $(X, \mathsf{q})$ *satisfy* (C0). *A set* $A \subseteq X$ *is* open *in* $(X, \mathsf{q})$ *if* $A = \mathsf{int}(A)$. *It is* closed *if* $A = \mathsf{cl}(A)$.
*A set* $N \subset X$ *is a* $\tau$-neighborhood *of* $x$ *if there is an open set* $O$ *(open neighborhood) such that* $x \in O \subseteq N$. *The* $\tau$*-neighborhoods of* $x$ *form a filter* $\mathcal{T}_{\mathsf{q}}(x)$, *which we call the* topological neighborhood filter *of* $X$.

Clearly, the collection of open sets that contain $x$ form a basis of the topological neighborhood filter $\mathcal{T}_{\mathsf{q}}(x)$. The open neighborhoods form a filter basis of $\mathcal{T}_{\mathsf{q}}(x)$.

**Lemma 1.** *(i) $\mathcal{T}_{\mathsf{q}}(x) \subseteq \mathcal{N}_{\mathsf{q}}(x)$. (ii) A set $A$ is open if and only if for each each $x \in A$ there is $N \in \mathcal{N}_{\mathsf{q}}(x)$ such that $N \subseteq A$. (iii) $\mathcal{T}_{\mathsf{q}}(x) = \mathcal{N}_{\mathsf{q}}(x)$ if and only if each neighborhood $F \in \mathcal{N}_{\mathsf{q}}(x)$ contains an open set $O$ with $x \in O$, i.e., if each neighborhood contains an open neighborhood.*

*Proof.* (i) follows directly from the definition of $\mathcal{T}_{\mathsf{q}}(x)$. (ii) $A$ is open iff for each $x \in A$ holds $A \in \mathcal{N}_{\mathsf{q}}(x)$. The filter axioms insure that this is equivalent to the seemingly weaker condition of the lemma. (iii) If $\mathcal{T}_{\mathsf{q}}(x) = \mathcal{N}_{\mathsf{q}}(x)$ if and only if every $A \in \mathcal{N}_{\mathsf{q}}(x)$ contains a basis element of $\mathcal{T}_{\mathsf{q}}(x)$, i.e., an open set $O$ that in turn is a neighborhood of $x$. Since $x \in O$ implies that there is $N \subseteq O$ with $x \in N$ and $N \in \mathcal{N}_{\mathsf{q}}(x)$, claim (iii) follows. $\square$

**Theorem 2.** *Let $(X, \mathsf{q})$ be a generalized convergence space. Then the sets $\mathcal{O}$ of all open sets and $\mathcal{C}$ of all closed satisfy*

- (O0) *A set $A$ is open if and only if its complement $X \setminus A$ is closed.*
- (O1) *$\emptyset \in \mathcal{O}$, $X \in \mathcal{O}$, and $\emptyset \in \mathcal{C}$, $X \in \mathcal{C}$.*
- (O2) *If $O_1, O_2 \in \mathcal{O}$ then $O_1 \cup O_2 \in \mathcal{O}$. If $C_1, C_2 \in \mathcal{C}$ then $C_1 \cap C_2 \in \mathcal{C}$.*
- (O3) *If $O_i \in \mathcal{O}$ for all $i \in I$, then $\bigcup\{O_i | i \in I\} \in \mathcal{O}$. If $C_i \in \mathcal{C}$ for all $i \in I$, then $\bigcap\{C_i | i \in I\} \in \mathcal{C}$.*

*Proof.* (O0) We have $A \in \mathcal{O} \iff A = \mathsf{int}(A)$. This is equivalent to $X \setminus A = X \setminus \mathsf{int}(A) = \mathsf{cl}(X \setminus A) \iff X \setminus A \in \mathcal{C}$.
(O1) follows immediately from (I) and (I').
(O2) follows directly from property (II') of the closure and interior operators.
(O3) Consider a collection $\{C_i | i \in I\}$ of closed sets, where $I$ is an arbitrary index set. Then we have $\bigcap_{j \in I} C_j \subseteq C_i$ for all $i \in I$. Property (II) implies $\mathsf{cl}\Big(\bigcap_{j \in I} C_j\Big) \subseteq \mathsf{cl}(C_i) = C_i$ for all $i \in I$ and hence $\mathsf{cl}\Big(\bigcap_{j \in I} C_j\Big) \subseteq \bigcap_{i \in I} C_i$. On the other hand, (III) implies $\bigcap_{i \in I} C_i \subseteq \mathsf{cl}\big(\bigcap_{i \in I} C_i\big)$. Therefore $\bigcap_{i \in I} C_i = \mathsf{cl}\big(\bigcap_{i \in I} C_i\big)$, i.e., $\bigcap_{i \in I} C_i \in \mathcal{C}$. The corresponding proposition for the union of open sets follows from (O0). $\square$

Note that $(X, \mathcal{O})$ forms a topological space in the conventional sense, see e.g. [9]. The neighborhood filters in this space are exactly the topological neighborhood filters $\mathcal{T}_{\mathsf{q}}(x)$. Hence, if $\mathcal{T}_{\mathsf{q}}(x) \neq \mathcal{N}_{\mathsf{q}}(x)$ we obtain a notion of neighborhood that is more general than the topological concept.

### 3.5. Convergence Spaces and Topology.

**Definition 7.** *Let $X$ be a set, $\Phi X$ the set of filters on $X$, and $\mathsf{q} \subseteq \Phi X \times X$ a relation. We say the filter $\mathcal{F} \in \Phi X$ converges to $x$ with respect to $\mathsf{q}$ if $(\mathcal{F}, x) \in \mathsf{q}$. For convenience we write $\mathcal{F} \to_{\mathsf{q}} x$. Consider the following axioms*

- (C2) *$\mathcal{F} \to_{\mathsf{q}} x$ implies $(\mathcal{F} \cap \dot{x}) \to_{\mathsf{q}} x$.*
- (C3) *$\mathcal{F} \to_{\mathsf{q}} x$ and $\mathcal{G} \to_{\mathsf{q}} x$ implies $(\mathcal{F} \cap \mathcal{G}) \to_{\mathsf{q}} x$.*
- (C4) *$\mathcal{F} \to_{\mathsf{q}} x$ whenever all ultrafilters $\mathcal{U}$ that are finer than $\mathcal{F}$ converge to $x$.*

(C5) *The neighborhood filters* $\mathcal{N}_{\mathsf{q}}(x)$ *converge to* $x$ *for all* $x \in X$.
(C6) *The topological neighborhood filters* $\mathcal{T}_{\mathsf{q}}(x)$ *converge to* $x$ *for all* $x \in X$.

*A generalized convergence space satisfying* (C2) *is a* Kent convergence space. *If it satisfies* (C3) *it is called a* limit space*, if* (C4) *holds, we have a* pseudotopological convergence space*, if* (C5) *holds we have a* pretopological convergence space*, and if* (C6) *holds, we have a* topological convergence space*. If* (S) *holds,* $(X, \mathsf{q})$ *is called* symmetric*.*

The following obvious implications hold for general convergence spaces:

topological $\Longrightarrow$ pretopological $\Longrightarrow$ pseudotopological $\Longrightarrow$ limit space $\Longrightarrow$ Kent

Kent convergence spaces are characterized by the fact that they can be represented as an infimum of a set of topological spaces. $(X, \mathsf{q})$ is a Kent convergence space if and only if there is a collection $\mathsf{Q}$ of topological convergence relations such that for all $\mathcal{F} \in \Phi X$ holds

$$\lim{}_{\mathsf{q}}\mathcal{F} = \sup_{\mathsf{q}' \in \mathsf{Q}} \lim{}_{\mathsf{q}'}\mathcal{F} \tag{8}$$

For the details see [17]. Limit spaces [8] and pseudotopological spaces were introduced by Fischer and Choquet, resp., as generalizations of topological spaces. They were further generalized by Kent [16]. Pretopological spaces, also introduced by Choquet [4], will be discussed in the following section.

Two topological convergence spaces will be of particular importance in the following. The *discrete convergence space* $(X, \mathsf{d})$ is defined by the fact that $\mathcal{F} \to_{\mathsf{d}} x$ implies $\mathcal{F} = \dot{x}$, i.e., $\mathcal{N}_{\mathsf{d}}(x) = \dot{x}$, and every set $A \subseteq X$ is both open and closed. In other words the discrete convergence structure $\mathsf{d}$ on $X$ gives rise to the discrete topology on $X$. In the *indiscrete convergence space* $(X, \mathsf{i})$ every filter converges to every point, i.e., $\mathcal{N}_{\mathsf{i}}(x) = X$ for all $x \in X$, and hence $X$ is the only non-empty open or closed set. Obviously, discrete and indiscrete convergence spaces are topological.

**Definition 8.** *Let* $(X, \mathsf{q}_1)$ *and* $(X, \mathsf{q}_2)$ *be generalized convergence spaces. We say that* $\mathsf{q}_1$ *is* finer *than* $\mathsf{q}_2$ *and* $\mathsf{q}_2$ *is* coarser *than* $\mathsf{q}_1$ *if* $\mathcal{F} \to_{\mathsf{q}_1} x$ *implies* $\mathcal{F} \to_{\mathsf{q}_2} x$ *for all* $x \in X$ *and* $\mathcal{F} \in \Phi X$. *We write* $\mathsf{q}_1 \subseteq \mathsf{q}_2$*, regarding the convergence relations* $\mathsf{q}_1$ *and* $\mathsf{q}_2$ *as subsets of* $X \times \Phi X$.

As an immediate consequence we see that

$$\mathsf{q}_1 \subseteq \mathsf{q}_2 \qquad \text{implies} \qquad \mathcal{N}_{\mathsf{q}_2}(x) \subseteq \mathcal{N}_{\mathsf{q}_1}(x), \tag{9}$$

since $\mathcal{N}_{\mathsf{q}_2}(x) = \cap\{\mathcal{F} | \mathcal{F} \to_{\mathsf{q}_2} x\} \subseteq \cap\{\mathcal{F} | \mathcal{F} \to_{\mathsf{q}_1} x\} = \mathcal{N}_{\mathsf{q}_1}(x)$.

Kent [16] shows that for each Kent convergence relation $\mathsf{q}$ there is a finest limit, pseudotopological, pretopological, and topological convergence relation that is coarser than $\mathsf{q}$. Their construction starts from $\mathsf{q}$ and proceeds by applying the appropriate axiom (C3) through (C6) in order to obtain additional converging filters.

**3.6. Pretopological Spaces.** Pretopological spaces play a crucial role in our discussion. The following theorem shows that these structure can be based on the notion of neighborhood.

**Theorem 3.** *Let $(X, \mathsf{q})$ be a pretopological convergence space, with closure and interior operators given by definition 5. Then we have*

$$\begin{aligned}\mathsf{int}(A) &= \{x \in X | A \in \mathcal{N}_{\mathsf{q}}(x)\} \\ \mathcal{N}_{\mathsf{q}}(x) &= \left\{F \subseteq X \middle| x \in \mathsf{int}(F)\right\} \\ \mathsf{cl}(A) &= \{x \in A | Q \cap A \neq \emptyset \textit{ for all } Q \in \mathcal{N}_{\mathsf{q}}(x)\}\end{aligned} \tag{10}$$

*Proof.* By definition, the neighborhood filter $\mathcal{N}_{\mathsf{q}}(x)$ is the intersection of all filters $\mathcal{F}$ that $\mathsf{q}$-converge to $x$. Thus $A \in \mathcal{N}_{\mathsf{q}}(x)$ implies that $A \in \mathcal{F}$ holds for all filters that converge to $x$. Conversely, if $A \in \mathcal{F}$ for all $\mathcal{F} \to_{\mathsf{q}} x$, then $A \in \mathcal{N}_{\mathsf{q}}(x)$ is true. Hence "$\mathcal{F} \to_{\mathsf{q}} x$ implies $A \in \mathcal{F}$" is equivalent to $A \in \mathcal{N}_{\mathsf{q}}(x)$. This proves the first equation. In order to verify the second line we observe that $x \in \mathsf{int}(F)$ is equivalent to $F \in \mathcal{F}$ for each filter that converges to $x$. Hence $F \in \mathcal{N}_{\mathsf{q}}(x)$. Conversely, if $F \in \mathcal{N}_{\mathsf{q}}(x)$ then $F \in \mathcal{F}$ for each filter the converges to $x$, i.e., $\mathcal{F} \to_{\mathsf{q}} x$ implies $F \in \mathcal{F}$, and thus $x \in \mathsf{int}(F)$.
The equation for $\mathsf{cl}(A)$ now follows from (0). □

Theorem 3 shows that neighborhood filters, closure operators (satisfying (I), (II) and (III)) and interior operators are equivalent. If one of these objects is known, the other two can be recovered from (0) and equ.(10), respectively. Instead of prescribing the convergence relation $\mathsf{q}$ of a pretopological space we may therefore start from the collection $\mathfrak{N} : x \mapsto \mathcal{N}(x)$, $x \in X$ of neighborhood filters which, for all $x \in X$ satisfy $x \in N_x$ for all $N_x \in \mathcal{N}(x)$. Given $(X, \mathfrak{N})$ we obtain the equivalent convergence space $(X, \mathsf{q}_{\mathfrak{N}})$ by *defining*

$$\mathcal{F} \to_{\mathsf{q}_{\mathfrak{N}}} x \iff \mathcal{N}(x) \subseteq \mathcal{F} \tag{11}$$

It is then easy to verify the following consistency result:

**Lemma 2.** $(X, \mathsf{q}_{\mathfrak{N}})$ *is a pretopological convergence space. For all $x \in X$ it satisfies $\mathcal{N}_{\mathsf{q}_{\mathfrak{N}}}(x) = \mathcal{N}(x)$.*

*Proof.* From $x \in N$ for all $N \in \mathcal{N}(x)$ we see $\mathcal{N}(x) \subset \dot{x}$ and hence (C1) holds. From $\mathcal{F} \subset \mathcal{G}$ and $\mathcal{F} \to_{\mathsf{q}_{\mathfrak{N}}} x$ follows $\mathcal{G} \to_{\mathsf{q}_{\mathfrak{N}}} x$ since $\mathcal{G}$ is finer than $\mathcal{F}$ and hence also finer than $\mathcal{N}(x)$; this verifies (C0). Finally, (C5) follows by definition since $\mathcal{N}(x) \to_{\mathsf{q}_{\mathfrak{N}}} x$ and hence $\mathcal{N}(x) = \bigcap\{\mathcal{F} | \mathcal{F} \to_{\mathsf{q}_{\mathfrak{N}}} x\} = \mathcal{N}_{\mathsf{q}_{\mathfrak{N}}}(x)$. □

In the following, and in the main text, we speak of pretopological spaces $(X, \mathfrak{N})$ instead of pretopological convergence spaces $(X, \mathsf{q}_{\mathfrak{N}})$ since prescribing the neighborhood filters will be more intuitive for our applications than prescribing the abstract convergence relation $\mathsf{q}$.

**Theorem 4.** *A pretopological space $(X, \mathfrak{N})$ is topological if and only if one, and therefore all, of the following equivalent conditions are satisfied*

*(i)* $\mathcal{N}(x) = \mathcal{T}(x)$ *for all* $x \in X$.
*(ii)* *For each $x \in X$ and each $N \in \mathcal{N}(x)$ there is a subset $A \subseteq N$ such that $x \in A$ and $N \in \mathcal{N}(y)$ for every $y \in A$.*
*(iii)* *The interior operator is idempotent, i.e.,* $\mathsf{int}(\mathsf{int}(A)) = \mathsf{int}(A)$.
*(iv)* *The closure operator is idempotent, i.e.,* $\mathsf{cl}(\mathsf{cl}(A)) = \mathsf{cl}(A)$.

*Proof.* (i$\Longleftrightarrow$ii) is e.g. [9, Thm.I.7.1], (ii$\Longleftrightarrow$iv) is [9, Thm.I.3.2], (iii$\Longleftrightarrow$iv) is easily obtained from equ.(10). $\square$

The comparison of convergence relations translates into the familiar partial order for pretopologies and topologies: If $(X,\mathfrak{N}_1)$ and $(X,\mathfrak{N}_2)$ are pretopological spaces, then $\mathfrak{N}_1$ is finer than $\mathfrak{N}_2$ if and only if $\mathcal{N}_2(x)\subseteq\mathcal{N}_1(x)$ for all $x\in X$. If $(X,\mathsf{q}_1)$ and $(X,\mathsf{q}_2)$ are topological convergence spaces, then $\mathsf{q}_1\subseteq\mathsf{q}_2$ if and only if $\mathcal{T}_2(x)\subseteq\mathcal{T}_1(x)$ for all $x\in X$. This is equivalent to the more familar characterization that $(X,\mathcal{O}_1)$ is finer than $(X,\mathcal{O}_2)$ if and only if $\mathcal{O}_2\subseteq\mathcal{O}_1$, i.e., if the open sets w.r.t. to the topology $\mathcal{O}_1$ are also open w.r.t. the topology $\mathcal{O}_2$.

The following result [16, Thm.1] easily follows from the discussion above.

**Lemma 3.** *Let $(X,\mathsf{q})$ be a generalized convergence space. Then*

$$\mathsf{q}\subseteq\mathsf{q}_{\mathfrak{N}_\mathsf{q}}\subseteq\mathsf{q}_{\mathcal{O}_\mathsf{q}} \tag{12}$$

*where $\mathsf{q}_{\mathfrak{N}_\mathsf{q}}$ and $\mathsf{q}_{\mathcal{O}_\mathsf{q}}$ denote the convergence relations of the pretopological space $(X,\mathfrak{N}_{\mathrm{II}})$ and the topological space $(X,\mathcal{O}_{\mathrm{II}})$, respectively.*

The following result will be useful for the discussion of continuous functions:

**Lemma 4.** *Let $(X,\mathsf{q}_1)$ and $(X,\mathsf{q}_2)$ be generalized convergence spaces such that $\mathsf{q}_1$ is finer than $\mathsf{q}_2$. Then $\mathfrak{N}_{\mathsf{q}_1}$ is finer than $\mathfrak{N}_{\mathsf{q}_2}$ and $\mathcal{O}_{\mathsf{q}_1}$ is finer than $\mathcal{O}_{\mathsf{q}_2}$.*

*Proof.* The first part follows immediately from equ.(9) above.
Let $O$ be an open neighborhood of $y$ w.r.t. $\mathsf{q}_2$. Thus for all $x\in O$, there is $N^x_{\mathsf{q}_2}$ such that $x\in N^x_{\mathsf{q}_2}\subseteq O$. From $\mathcal{N}_{\mathsf{q}_2}(x)\subseteq\mathcal{N}_{\mathsf{q}_1}(x)$ we know that for each $N^x_{\mathsf{q}_2}$ there is $x\in N^x_{\mathsf{q}_1}\subseteq N^x_{\mathsf{q}_2}\subseteq O$, and hence $O$ is open w.r.t. $\mathcal{O}_{\mathsf{q}_1}$ as well. $\mathcal{O}_{\mathsf{q}_2}\subseteq\mathcal{O}_{\mathsf{q}_1}$. $\square$

**3.7. Composition of Pretopologies.** A useful composition of pretopologies on a set $X$ is described in [7]. Let $\mathsf{p}$ and $\mathsf{q}$ be two pretopologies on $X$ with neighborhood filters $\mathcal{N}_\mathsf{p}(x)$ and $\mathcal{N}_\mathsf{q}(x)$. We set

$$\mathcal{N}_{\mathsf{pq}}(x)=\{A\subseteq X|\mathrm{int}^\mathsf{p}A\in\mathcal{N}_\mathsf{q}(x)\} \tag{13}$$

where $\mathrm{int}^\mathsf{p}$ denotes the interior operator w.r.t. to the pretopology p. The closure and interior operators satisfy

$$\mathrm{int}^{\mathsf{pq}}=\mathrm{int}^\mathsf{q}\mathrm{int}^\mathsf{p}\quad\text{ and }\quad\mathrm{cl}^{\mathsf{pq}}=\mathrm{cl}^\mathsf{q}\mathrm{cl}^\mathsf{p} \tag{14}$$

The discrete topology $\mathsf{d}$ is the unit element of this composition, while the indiscrete topology $\mathsf{i}$ acts as null element, i.e., $\mathsf{pd}=\mathsf{dp}=\mathsf{p}$ and $\mathsf{pi}=\mathsf{ip}=\mathsf{i}$.

Given two pretopologies $\mathsf{p}$ and $\mathsf{q}$ we define $\mathsf{p}\wedge\mathsf{q}$ via its neighborhood filters

$$\mathcal{N}_{\mathsf{p}\wedge\mathsf{q}}(x)=\mathcal{N}_\mathsf{q}(x)\cup\mathcal{N}_\mathsf{q}(x) \tag{15}$$

Let $\alpha$ be an ordinal number. Then define $\mathsf{p}^\alpha=\mathsf{p}^{\alpha-1}\mathsf{p}$ if $\mathsf{p}^{\alpha-1}$ exists and $\mathsf{p}^\alpha=\bigwedge_{\beta<\alpha}\mathsf{p}^\alpha$ otherwise. The powers of $\mathsf{p}$ satisfy $\mathsf{p}^{\alpha+\beta}=\mathsf{p}^\alpha\mathsf{p}^\beta$ and $(\mathsf{p}^\alpha)^\beta=\mathsf{p}^{\alpha\beta}$ for all ordinal numbers. For each $\mathsf{p}$ there is least ordinal number $t(\mathsf{p})$, called the topological defect, such that $\mathsf{p}^{t(\mathsf{p})}$ is a topology [3, 11, 12, 22, 24]. In the following we shall write $\mathsf{cl}(A)^\alpha$ for the closure of $A$ in $\mathsf{p}^\alpha$. The "topological closure" is $\widehat{A}=\mathsf{cl}(A)^{t(\mathsf{p})}$.

## 4. Uniform Structures

### 4.1. Symmetry.

(S') $x \in \mathcal{N}(y)\downarrow$ implies $y \in \mathcal{N}(x)\downarrow$.
(S) $\mathcal{F} \to_{\mathsf{q}} x$ and $y \in \mathcal{F}\downarrow$ implies $\mathcal{F} \to_{\mathsf{q}} y$.
(R0) $x \in \mathsf{cl}(y)$ implies $y \in \mathsf{cl}(x)$ for all $x, y \in X$.

In a topological space (S) and (R0) are equivalent, see e.g. [28]. In pretopological spaces (S') and (R0) are equivalent.

If $(X, \mathsf{q})$ is a pretopological space, then (S) trivially implies the much weaker symmetry condition (S'). We remark that Kent [18] calls (S') symmetric while Preuß [27] uses "symmetric" for (S). In [17] it is shown that (S) is equivalent to "weak uniformizability", meaning that $\mathsf{q}$ can be represented as the infimum of a set of completely regular topologies on $X$ analogous to equ.(8).

**Lemma 5.** *A pretopological space satisfies* (S) *if and only if* $y \in \mathcal{N}(x)\downarrow$ *implies* $\mathcal{N}(y) = \mathcal{N}(x)$.

*Proof.* (See also [17, Thm.2.2.] Substituting $\mathcal{N}(x)$ for $\mathcal{F}$ in (S) we find that in a (S) pretopological space $y \in \mathcal{N}(x)\downarrow$ implies $\mathcal{N}(x) \to y$ and hence $\mathcal{N}(y) \subseteq \mathcal{N}(x)$. Consequently, $x \in \mathcal{N}(x)\downarrow \subseteq \mathcal{N}(y)\downarrow$. Thus, $\mathcal{N}(y) \to y$ and (S) imply $\mathcal{N}(y) \to x$, whence $\mathcal{N}(y) \subseteq \mathcal{N}(x)$. Therefore $\mathcal{N}(y) = \mathcal{N}(x)$
Conversely, suppose $y \in \mathcal{N}(x)\downarrow$ implies $\mathcal{N}(y) = \mathcal{N}(x)$. Since $\mathcal{F} \to x$ means $\mathcal{N}(x) \subseteq \mathcal{F}$ we have $\mathcal{F}\downarrow \subseteq \mathcal{N}(x)\downarrow$ and $y \in \mathcal{F}\downarrow$ implies $y \in \mathcal{N}(x)\downarrow$; now $\mathcal{N}(y) = \mathcal{N}(x)$ guarantees $\mathcal{F} \to y$. ☐

**4.2. Uniformities and Filters on $X \times X$.** We first recall the definition of the relation product of subsets of $X \times X$:

$$\begin{aligned} \Delta &= \{(x,x) | x \in X\} \\ F^{-1} &= \{(x,y) | (y,x) \in F\} \\ F \circ G &= \big\{(x,y) \big| \exists\, z \in X : (x,z) \in G \text{ and } (z,y) \in F\big\} \\ F[x] &= \{y \in X | (x,y) \in F\} \end{aligned} \tag{16}$$

Note that the relation product $\circ$ is associative, satisfies $\Delta \circ F = F \circ \Delta = F$ and $(F^{-1})^{-1} = F$.

Now let $\mathcal{F}, \mathcal{G} \in \Phi[X \times X]$. We define

$$\begin{aligned} \dot{x} \times \dot{x} &= \{A \subset X \times X | (x,x) \in A\} \\ \dot{\Delta} &= \{A \subset X \times X | \Delta \subseteq A\} \\ \mathcal{F}^{-1} &= \{F^{-1} | F \in \mathcal{F}\} \\ \mathcal{F} \circ \mathcal{G} &= \begin{cases} \{H | \exists F \in \mathcal{F}, G \in \mathcal{G} : F \circ G \subseteq H\} & \text{if} \quad F \circ G \neq \emptyset\, \forall\, F \in \mathcal{F}, G \in \mathcal{G} \\ \emptyset & \text{otherwise} \end{cases} \\ \mathcal{F} \times \mathcal{G} &= \{H | \exists F \in \mathcal{F} : F \times G \subseteq H\} \\ \mathcal{F}[x] &= \{F[x] \subset X | F \in \mathcal{F}\} \end{aligned} \tag{17}$$

Let $\mathcal{U} \in \Phi[X \times X]$ and consider the following properties.

(u1) $\Delta \subseteq U$ for all $U \in \mathcal{U}$.
(u2) $U \in \mathcal{U}$ implies $U^{-1} \in \mathcal{U}$.
(u3) For all $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V \circ V \subset U$.
(u4) For all $U \in \mathcal{U}$ and all $x \in X$ there is $V \in \mathcal{U}$ such that $(V \circ V)[x] \subset U[x]$.

The filter $\mathcal{U}$ is called a *preuniformity* on $X$ if (u1) is satisfied, a *semiuniformity* if (u1) and (u2) hold, a *quasiuniformity* if (u1) and (u3) hold, a *uniformity* if (u1), (u2), and (u3) hold, and a *local uniformity* if (u1), (u2), and (u4) hold. The elements $U \in \mathcal{U}$ are called *entourages*, see e.g. [25]

### 4.3. Preuniform Convergence Spaces.

**Definition 9.** *Let $X$ be a set and let $\Upsilon \subseteq \Phi(X \times X)$ be a set of filters on $X \times X$. Consider the following properties:*

(U0) $\mathcal{G} \in \Upsilon$ *and* $\mathcal{G} \subseteq \mathcal{F}$ *implies* $\mathcal{F} \in \Upsilon$.
(U1) $\dot{x} \times \dot{x} \in \Upsilon$ *for all* $x \in X$.
(UD) $\dot{\Delta} \in \Upsilon$.
(US) $\mathcal{F} \in \Upsilon$ *implies* $\mathcal{F}^{-1} \in \Upsilon$.
(U3) $\mathsf{F}, \mathsf{G} \in \Upsilon$ *implies* $\mathsf{F} \cap \mathsf{G} \in \Upsilon$.
(UP) *There is a filter* $\mathcal{U} \in \Phi[X \times X]$ *such that* $\Upsilon = \{\mathcal{F} \in \Phi[X \times X] | \mathcal{U} \subseteq \mathcal{F}\}$.
(U4) $\mathsf{F}, \mathsf{G} \in \Upsilon$ *and* $\mathsf{F} \circ \mathsf{G} \neq \emptyset$ *implies* $\mathsf{F} \circ \mathsf{G} \in \Upsilon$.

*The pair $(X, \Upsilon)$ is a* preuniform convergence space *if it satisfies* (U0) *and* (U1) [1]. *Note that $\dot{\Delta} \subseteq \dot{x} \times \dot{x}$ for all $x \in X$. Hence* (UD) *and* (U0) *imply* (U1). *A preuniform convergence space satisfying* (UD), (UP), (US) *is* diagonal, principal, *a* semiuniform convergence space, *respectively. A semiuniform convergence space satisfying* (U3) *is a a* semi-uniform limit space. *Note that* (UP) *implies* (U3). *Principal convergence spaces can be identified with the uniformities discussed in the previous subsection. A semiuniform limit space is called a* uniform limit space *if it satisfies* (U4). *These were first studied by* [30]. *Principal uniform limit space are equivalent to Weil's* [29] *notion of "uniform spaces".*

Let $(X, \Upsilon)$ be a pre-uniform convergence space. Then $(X, \mathsf{q}_\Upsilon)$ defined by

$$\mathcal{H} \to_{\mathsf{q}_\Upsilon} x \text{ whenever } \exists \mathcal{F} \in \Upsilon \text{such that } \mathcal{F}[x] \subseteq \mathcal{H} \tag{18}$$

Since $(\dot{x} \times \dot{x})[x] = \dot{x}$, we see that (C1) follows from (U1), i.e., $(X, \mathsf{q}_\Upsilon)$ is the generalized convergence space induced by $(X, \Upsilon)$.

Conversely, given a generalized convergence space $(X, \mathsf{q})$ we may construct

$$\Upsilon_{\mathsf{q}} = \left\{\mathcal{F} \in \Phi[X \times X] \middle| \exists \mathcal{H} \in \Phi X : \mathcal{H} \to_{\mathsf{q}} x \text{ and } (\dot{x} \times \mathcal{H}) \subseteq \mathcal{F}\right\} \tag{19}$$

It is clear that $(X, \Upsilon_{\mathsf{q}})$ is a preuniform convergence space.

It remains to show that these definitions are "consistent"

**Lemma 6.** $\mathsf{q}_{\Upsilon_{\mathsf{q}}} = \mathsf{q}$.

*Proof.*

$$\begin{aligned}\mathcal{H} \to_{\mathsf{q}_{\Upsilon_\mathsf{q}}} x &\iff \exists \mathcal{F} \text{ such that } \mathcal{F}[x] \subseteq \mathcal{H} \text{ and } \exists \mathcal{H}' \text{ such that } \mathcal{H}' \to_{\mathsf{q}} x (\dot{x} \times \mathcal{H}') \subseteq \mathcal{F} \\ &\iff \exists \mathcal{F} \text{ such that } \exists \mathcal{H}' : \mathcal{H}' \to_{\mathsf{q}} x \text{ and } \mathcal{H}' = (\dot{x} \times \mathcal{H}')[x] \subseteq \mathcal{F}[x] \subseteq \mathcal{H} \\ &\iff \exists \mathcal{H}' : \mathcal{H}' \to_{\mathsf{q}} x \text{ and } \mathcal{H}' \subseteq \mathcal{H} \\ &\iff \mathcal{H} \to_{\mathsf{q}} x\end{aligned}$$

□

A semiuniform convergence space $(X, \Upsilon)$ has an underlying Kent convergence space $(X, \mathsf{q}_\Upsilon)$ defined by

$$\mathcal{F} \to_{\mathsf{q}_\Upsilon} x \qquad \iff \qquad (\mathcal{F} \cap \dot{x}) \times (\mathcal{F} \cap \dot{x}) \in \Upsilon \tag{20}$$

which satisfies (S). Note that equ.(20) follows immediately from equ.(18), axiom (US), and axiom (C2). Conversely, every Kent convergence space satisfying (S) may be considered as a semiuniform convergence space $(X, \Upsilon_\mathsf{q})$ where

$$\Upsilon_\mathsf{q} = \left\{\mathcal{F} \,\middle|\, \exists \mathcal{G} \in \Phi X \text{ and } x \in X \text{ such that } \mathcal{G} \to_\mathsf{q} x \text{ and } \mathcal{G} \times \mathcal{G} \subseteq F\right\} \tag{21}$$

For more details we refer to [27].

Let $(X, \Upsilon)$ be a semiuniform convergence space, and consider the following axioms:

There exists a substantial body of literature on various types of uniformizations of convergence spaces, see e.g. [2, 6, 19, 15, 27].

## 5. Continuity

**5.1. Continuity in Convergence Spaces.** Let $f : X \to Y$ be an arbitrary function and let $\mathcal{F}$ be a filter on $X$. Then we define $f(\mathcal{F}) = \{f(F) | F \in \mathcal{F}\}$. It is easy to see that $f(\mathcal{F})$ is a filter basis: Since $\emptyset \notin \mathcal{F}$ it follows that $f(F) \neq \emptyset$. Property (F2) follows immediately from $f(F \cap F') \subseteq f(F) \cap f(F')$ for any two sets $F, F'$.

**Definition 10.** *Let $(X, \mathsf{q})$ and $(Y, \mathsf{p})$ be two generalized convergence spaces and let $f : X \to Y$ be a function. Then $f$ is* continuous *in $x \in X$ (w.r.t. the relations $\mathsf{q}$ and $\mathsf{p}$) if*

$$\mathcal{F} \to_\mathsf{q} x \quad \textit{implies} \quad f(\mathcal{F}) \to_\mathsf{p} f(x) \tag{22}$$

*The function $f$ is* continuous *if it is continuous for all $x \in X$.*

**Lemma 7.** *Let $(X, \mathfrak{N})$ and $(Y, \mathfrak{M})$ be two pretopological spaces. Then $f : X \to Y$ is continuous in $x \in X$ (w.r.t. the associated convergence relations $\mathsf{q} = \mathsf{q}_\mathfrak{N}$ and $\mathsf{p} = \mathsf{p}_\mathfrak{M}$) if*

$$\textit{For each } M \in \mathcal{M}(f(x)) \textit{ there is } N \in \mathcal{N}(x) \textit{ such that } f(N) \subseteq M. \tag{23}$$

*Proof.* Definition 10 translates to: $f$ is continuous in $x$ if and only if $\mathcal{N}(x) \subseteq \mathcal{F}$ implies $\mathcal{M}(f(x)) \subseteq f(\mathcal{F})\uparrow$. This implies in turn

$$f : X \to Y \text{ is continuous in } x \text{ if and only if } \mathcal{M}(f(x)) \subseteq f(\mathcal{N}(x))\uparrow \tag{24}$$

since $\mathcal{F} \subseteq \mathcal{F}'$ implies $f(\mathcal{F}) \subseteq f(\mathcal{F}')$ and hence $f(\mathcal{F})\uparrow \subseteq f(\mathcal{F}')\uparrow$.

It remains to show that (24) and (23) are equivalent. If condition (24) holds, then for each $M \in \mathcal{M}(f(x))$ there is a set $M' \subseteq M$ such that $M' \in f(\mathcal{N}(x))$, and hence there is an $N \in \mathcal{N}(x)$ such that $M' = f(N)$, i.e., equ.(23) holds. Conversely, if for each $M \in \mathcal{M}(f(x))$ there is an $N \in \mathcal{N}(x)$ such that $f(N) \subset M$, then $M \in f(\mathcal{N}(x))$, i.e., $f(\mathcal{N}(x))\uparrow$ is finer than $\mathcal{M}(f(x))$. $\square$

The notion of convergence in pretopological spaces coincides with the usual definitions of convergence in topological spaces if $(X, \mathsf{q})$, or $(X, \mathfrak{N})$, is topological.

**Theorem 5.** *Consider a function $f : X \to Y$ Then:*

*(i) If $\mathsf{q}_2 \subseteq \mathsf{q}_1$ and $f : (X, \mathsf{q}_1) \to (Y, \mathsf{p})$ is continuous then $f : (X, \mathsf{q}_2) \to (Y, \mathsf{p})$ is also continuous.*
*(ii) If $\mathsf{p}_1 \subseteq \mathsf{p}_2$ and $f : (X, \mathsf{q}) \to (Y, \mathsf{p}_1)$ is continuous then $f : (X, \mathsf{q}) \to (Y, \mathsf{p}_2)$ is also continuous.*

*Proof.* (i) Suppose $f : (X, \mathsf{q}_1) \to (Y, \mathsf{p})$ is continuous. Then $(\mathcal{F}, x) \in \mathsf{q}_2 \subseteq \mathsf{q}_1$ implies $(f(\mathcal{F})\uparrow, f(x)) \in \mathsf{p}$, i.e., $f : (X, \mathsf{q}_2) \to (Y, \mathsf{p})$ is also continuous.
(ii) Suppose $f : (X, \mathsf{q}) \to (Y, \mathsf{p}_1)$ is continuous. Then $(\mathcal{F}, x) \in \mathsf{q}$ implies $(f(\mathcal{F})\uparrow, f(x)) \in \mathsf{p}_1 \subseteq \mathsf{p}_2$, i.e., $f : (X, \mathsf{q}) \to (Y, \mathsf{p}_2)$ is continuous. $\square$

**5.2. Final Convergence Relations.** Let $(X, \mathsf{q})$ be a generalized convergence space, let $Y$ be a set and $f : X \to Y$ a function. We may define a convergence relation {(II) by defining $\mathcal{G} \to_{\mathsf{f}(\mathsf{q})} y$ whenever there is coarser filter $\mathcal{H} = f(\mathcal{F})\uparrow$ such that $\mathcal{F} \to_{\mathsf{q}} x$ and $y = f(x)$. It is clear that $(Y, \mathsf{f}(\mathsf{q}))$ is a generalized convergence space, since $f(\dot{x})\uparrow = \dot{y}$ with $y = f(x)$. The convergence structure $\mathsf{f}(\mathsf{q})$ is called *final*. It is the coarsest convergence structure such that $f : (X, \mathsf{q}) \to Y$ is continuous [8].

Similarly, we may start with a pretopological convergence space $(X, \mathsf{q})$ and ask for the coarsest pretopology on on $Y$ such that $f$ is continuous. Note that this is in general not $\mathsf{f}(\mathsf{q})$), since in order to obtain a pretopology on $Y$ we need to require in addition that the intersection of any set of filters that converge to $x$ must also converge to $x$. Hence the *final pretopology* $\mathsf{fp}(\mathsf{q})$) has the neighborhood filters

$$\mathcal{A}(y) = \mathcal{N}_{\mathsf{fp}(\mathsf{q})}(y) = \bigcap_{x \in f^{-1}(y)} f\big(\mathcal{N}_{\mathsf{q}}(x)\big)\uparrow \tag{25}$$

if $f^{-1} \neq \emptyset$ and $\mathcal{A}(y) = \dot{y}$ otherwise. It is clear that $f : (X, \mathsf{q}) \to (X, \mathsf{fp}(\mathsf{q}))$ is continuous, see also [8].

We may use proposition 5 of [20] as definition: A continuous function $f : (X, \mathsf{q}) \to f : (Y, \mathsf{p})$ is a *pretopological quotient map* if and only if for each $y \in Y$ there is $x \in X$ such that $f(\mathcal{N}_{\mathsf{q}}(x))\uparrow = \mathcal{A}(y)$.

Consequently $\mathcal{A}(y) = \mathcal{S}(y)$, i.e., $f(\mathcal{N}_{\mathsf{q}}(x))\uparrow = \mathcal{A}(y)$ for all $x \in f^{-1}$ is a sufficient condition. Such quotient maps are called *neighborhood preserving.*

**5.3. Uniform Continuity.**

**Definition 11.** *A function $f : (X, \Upsilon) \to (Y, \Psi)$ from one preuniform convergence space into another one is* uniformly continuous *if*

$$(f \times f)(\mathcal{F}) \in \Psi \tag{26}$$

*for all $\mathcal{F}$ in $\Upsilon$.*

**Lemma 8.** *If $f : (X, \Upsilon) \to (Y, \Psi)$ is uniformly continuous, then $f : (X, \mathsf{q}_\Upsilon) \to (Y, \mathsf{q}_\Psi)$ is continuous.*

*Proof.* Suppose $\mathcal{H} \to_{\mathsf{q}_\Upsilon} x$, i.e., $\mathcal{F}[x] \subseteq \mathcal{H}$ for some $\mathcal{F} \in \Upsilon$, and hence $\dot{x} \times \mathcal{H} \in \Upsilon$. Assuming uniform continuity we have $(f \times f)(\dot{x} \times \mathcal{H}) \in \Psi$ and hence by definition $(f \times f)(\dot{x} \times \mathcal{H})[f(x)] \to_{\mathsf{q}_\Psi} f(x)$. It remains to compute

$$(f \times f)(\dot{x} \times \mathcal{H})[f(x)] = (f(\dot{x}) \times f(\mathcal{H}))[f(x)] = (\dot{f}(x) \times f(\mathcal{H}))[f(x)] = f(\mathcal{H})$$

and hence we have $f\mathcal{H}) \to_{\mathsf{q}_\Psi} f(x)$, i.e. $f$ is continuous. □

## 6. Connectedness

**Definition 12.** *Two sets $A, B \subseteq X$ are* separated *in $(X, \mathsf{q})$ if $\mathsf{cl}(A) \cap B = A \cap \mathsf{cl}(B) = \emptyset$.*

**Theorem 6.** *Let $(X, \mathsf{q})$ be a generalized convergence space. Then the following propositions are equivalent:*

(c1) *There is no proper subset of $X$ that is both open and closed.*
(c2) *$X$ cannot be represented as the union of two disjoint open sets.*
(c3) *$X$ cannot be represented as the union of two disjoint closed sets.*
(c4) *$X$ cannot be represented as the union of two separated sets.*

*Proof.* The equivalence of (c1), (c2), (c3) is obvious.
In order to show (c3)$\Longleftrightarrow$(c4) we first assume that $A$ and $X \setminus A$ are separated for some $A \neq \emptyset, X$. If $A$ is not closed, there is $x \in \mathsf{cl}(A) \setminus A$. This leads to a contradiction since $\mathsf{cl}(A) \cap (X \setminus A) = \emptyset$ implies $x \notin (X \setminus A)$, i.e., $x \in A$. Thus $A$ is closed. The same argument can be used to show that $X \setminus A$ is closed. Conversely, suppose $A$ and $X \setminus A$ are both closed. Then $A \cap (X \setminus A) = \mathsf{cl}(A) \cap (X \setminus A) = A \cap \mathsf{cl}(X \setminus A) = \emptyset$, i.e., $A$ and $X \setminus A$ are separated. □

**Corollary 1.** *A generalized convergence space $(X, \mathsf{q})$ is connected if and only if the associated topological space $(X, \mathcal{O}_\mathsf{q})$ is connected.*

**Definition 13.** *A generalized convergence space is* connected *and one, and hence all, of the conditions (c1) through (c4) are satisfied.*
*A generalized convergence space $(X, \mathsf{q})$ is* path connected *if for all $x, y \in X$ there a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. Such a function $f$ is called a* path.

**Theorem 7.** *If $(X, \mathsf{q})$ is path-connected, then it is connected.*

*Proof.* Let $(X, \mathcal{O}_\mathsf{q})$ be the topological space associated with $(X, \mathsf{q})$. If $f : [0, 1] \to (X, \mathsf{q})$ is is continuous, then, by Lemma 3 and theorem 5 $f : [0, 1] \to (X, \mathcal{O}\mathsf{q})$ is also continuous. Thus path-connectedness implies path-connectedness w.r.t. the associated topology $\mathcal{O}_\mathsf{q}$, which implies that $(X, \mathcal{O}_\mathsf{q})$ is connected. Corollary 1 completes the proof. □

## 7. Separation Properties

**7.1. "Lower" Separation Axioms.** A convergence space is

- $(\mathsf{T0})$ $\dot{x} \to_{\mathsf{q}} y$ and $\dot{y} \to_{\mathsf{q}} x$ implies $x = y$.
- $(\mathsf{T1})$ $\dot{x} \to_{\mathsf{q}} y$ implies $x = y$. In other words, if $x \neq y$ then $\dot{x} \not\to_{\mathsf{q}} y$.
- $(\mathsf{Re})$ *reciprocal* if $x, y \in \lim \mathcal{F}$ implies $\{\mathcal{F} | \mathcal{F} \to_{\mathsf{q}} x\} = \{\mathcal{G} | \mathcal{G} \to_{\mathsf{q}} y\}$.
- $(\mathsf{H})$ *Hausdorff* if $x, y \in \lim \mathcal{F}$ implies $x = y$, i.e., each filter converges to at most one point.
- $(\mathsf{T2})$ *separable* if $\mathcal{F} \to_{\mathsf{q}} x$, and $\mathcal{G} \to_{\mathsf{q}} y$, $x \neq y$ implies $\mathcal{F} \vee \mathcal{G} = \emptyset$.
- $(\alpha\mathsf{T2}\frac{1}{2})$ *$\alpha$-Urysohn* if $\mathcal{F} \to_{\mathsf{q}} x$, $\mathcal{G} \to_{\mathsf{q}} y$, $x \neq y$ implies $\mathsf{cl}(\mathcal{F})^{\alpha} \vee \mathsf{cl}(\mathcal{G})^{\alpha} = \emptyset$.
- $(\mathsf{wT2}\frac{1}{2})$ *Urysohn* if $\mathcal{F} \to_{\mathsf{q}} x$, $\mathcal{G} \to_{\mathsf{q}} y$, and $x \neq y$ implies $\mathsf{cl}(\mathcal{F}) \vee \mathsf{cl}(\mathcal{G}) = \emptyset$.
- $(\mathsf{T2}\frac{1}{2})$ $\aleph_0$-*Urysohn* if $\mathcal{F} \to_{\mathsf{q}} x$, $\mathcal{G} \to_{\mathsf{q}} y$, and $x \neq y$ implies $\mathsf{cl}(\mathcal{F})^{\aleph_0} \vee \mathsf{cl}(\mathcal{G})^{\aleph_0} = \emptyset$.
- $(\mathsf{T2}\frac{1}{2})$ *strongly Urysohn* if $\mathcal{F} \to_{\mathsf{q}} x$, $\mathcal{G} \to_{\mathsf{q}} y$, and $x \neq y$ implies $\widehat{\mathcal{F}} \vee \widehat{\mathcal{G}} = \emptyset$.

*Remark.* Some authors define $(\mathsf{T2})$ via the existence of disjoint open neighborhoods for any two points, e.g. [8].

The terms reciprocal was introduced in [23]. In [15] the property was introduced as "axiom P".

**Lemma 9.** *$(\mathsf{T2})$, $(\mathsf{H})$, $(\mathsf{T0}\wedge\mathsf{Re})$ are equivalent in any generalized convergence space $(X, \mathsf{q})$.*

*Proof.* If $(X, \mathsf{q})$ is not Hausdorff, then there are two points $x \neq y$ and a filter $\mathcal{F}$ such that $\mathcal{F} \to_{\mathsf{q}} x$ and $\mathcal{F} \to_{\mathsf{q}} y$. Since $\mathcal{F} \vee \mathcal{F} = \mathcal{F} \neq \emptyset$, $(X, \mathsf{q})$ is not separable.
If $(X, \mathsf{q})$ is not separable then there two points $x \neq y$ and two are non-disjoint filters $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{F} \to_{\mathsf{q}} x$ and $\mathcal{G} \to_{\mathsf{q}} y$. In this case $\mathcal{F} \vee \mathcal{G} \neq \emptyset$ $\mathsf{q}$-converges to both $x$ and $y$, violating $(\mathsf{H})$.
It is clear that $(\mathsf{H})$ implies $(\mathsf{Re})$ and $(\mathsf{T0})$. Suppose $\mathcal{F} \to_{\mathsf{q}} x$ and $\mathcal{F} \to_{\mathsf{q}} y$. By $(\mathsf{Re})$ we have $\{\mathcal{F} | \mathcal{F} \to_{\mathsf{q}} x\} = \{\mathcal{G} | \mathcal{F} \to_{\mathsf{q}} y\}$ and hence $\dot{x} \to_{\mathsf{q}} y$ and $\dot{y} \to_{\mathsf{q}} x$. $(\mathsf{T0})$ implies that $x = y$, thus every filter converges to at most one limit point, i.e., $(\mathsf{H})$ holds. ☐

$(\mathsf{Re})$ implies $(\mathsf{S})$.

**7.2. Regularity.** Remark. In many (older) references the notions regular and $(\mathsf{T3})$, normal and $(\mathsf{T4})$ are reversed.

Let $\alpha$ be an ordinal number, let $\mathcal{F}$ be a filter on $X$. Then we define:

$$\begin{aligned} \mathsf{cl}(\mathcal{F}) &= \{\mathsf{cl}(F) \,| F \in \mathcal{F}\}\!\uparrow \\ \mathsf{cl}(\mathcal{F})^{\alpha} &= \{\mathsf{cl}(F)^{\alpha} \, F \in \mathcal{F}\}\!\uparrow \\ \widetilde{\mathcal{F}} &= \{\mathsf{cl}(F)^{\aleph_0} \, F \in \mathcal{F}\}\!\uparrow \\ \widehat{\mathcal{F}} &= \{\widehat{F} | F \in \mathcal{F}\}\!\uparrow \end{aligned} \tag{27}$$

Note that $\mathsf{cl}(\mathcal{F})^{\alpha} \subseteq \mathsf{cl}(\mathcal{F})^{\beta}$ if $\alpha \geq \beta$, and hence $\widehat{\mathcal{F}} \subseteq \widetilde{\mathcal{F}} \subseteq \mathsf{cl}(\mathcal{F})$.

A generalized convergence space $(X, \mathsf{q})$ is

- $(\alpha\mathsf{R})$ $\alpha$-regular if $\mathcal{F} \to_{\mathsf{q}} x$ implies $\mathsf{cl}(\mathcal{F})^{\alpha} \to_{\mathsf{q}} x$;

(R) regular if $\mathcal{F} \to_{\mathsf{q}} x$ implies $\mathsf{cl}(\mathcal{F}) \to_{\mathsf{q}} x$;
($\aleph_0$R) $\aleph_0$-regular if $\mathcal{F} \to_{\mathsf{q}} x$ implies $\widetilde{\mathcal{F}} \to_{\mathsf{q}} x$;
(sR) strongly regular if $\mathcal{F} \to_{\mathsf{q}} x$ implies $\widehat{\mathcal{F}} \to_{\mathsf{q}} x$.

This terminology is consistent with [8, 13, 5, 21]. However, we use *regular* for what is sometimes called weakly-regular and $\aleph_0$-regular for regular in [21]. If $(X, \mathsf{q})$ is topological, then ($\alpha$R) is equivalent for all $\alpha$ since $\mathsf{cl}(\mathcal{F}) = \widehat{\mathcal{F}}$. Regularity coicides with the usual notion on topological spaces.

**Lemma 10.** *$\mathcal{F} \to_{\mathsf{q}} x$ implies $\mathsf{cl}(\mathcal{F}) \subseteq \dot{x}$.*

*Proof.* Consider $F \in \mathcal{F}$. By definition, $z \in \mathsf{cl}(F)$ iff there is a filter $\mathcal{G}$ such that $F \in \mathcal{G}$ and $\mathcal{G} \to_{\mathsf{q}} x$. Consequently, we have $x \in \mathsf{cl}(F)$. By construction we have for all $F' \in \mathsf{cl}(\mathcal{F})$: $x \in F'$ and hence $F' \in \dot{x}$, i.e., $\mathsf{cl}(\mathcal{F}) \subseteq \dot{x}$. □

A generalized convergence space $(X, \mathsf{q})$ is ($\alpha$-T3), (T3), ($\aleph_0$T3), or (sT3) if it satisfies the corresponding regularity axiom ($\alpha$R), (R), ($\aleph_0$R), or (sR), respectively and the separation axiom (T0). Such spaces are considered in some detail in [21].

**Lemma 11.** ($\alpha$-T3) *implies* ($\alpha$-T$2\frac{1}{2}$).

*Proof.* We proceed in two steps. First we show that a regular (T0) space is Hausdorff: To this end suppose $(X, \mathsf{q})$ is not separable, i.e., there are two distinct points $x \neq y$ and non-disjoint filters $\mathcal{F} \to_{\mathsf{q}} x$ and $\mathcal{G} \to_{\mathsf{q}} y$, i.e., $\mathcal{F} \vee \mathcal{G}$ q-converges to both $x$ and $y$. Hence, by lemma 10, $\mathsf{cl}(\mathcal{F} \vee \mathcal{G}) \subseteq \dot{x}$ and $\mathsf{cl}(\mathcal{F} \vee \mathcal{G}) \subseteq \dot{y}$. Now (R) implies that $\mathsf{cl}(\mathcal{F} \vee \mathcal{G})$ also q-converges to both $x$ and $y$. The same holds for the finer filters $\dot{x}$ and *doty*, contradicting (T0). A regular (T0) space hence is Hausdorff.
Now suppose $(X, \mathsf{q})$ is $\alpha$-regular and there are two points $x \neq y$ and filters $\mathcal{F} \to_{\mathsf{q}} x$, $\mathcal{G} \to_{\mathsf{q}} x$, such that $\mathsf{cl}(\mathcal{F})^{\alpha} \vee \mathsf{cl}(\mathcal{G})^{\alpha} \neq \emptyset$, i.e., ($\alpha$T$2\frac{1}{2}$) does not hold. Axiom ($\alpha$R) implies $\mathsf{cl}(\mathcal{F})^{\alpha} \to_{\mathsf{q}} x$ and $\mathsf{cl}(\mathcal{F})^{\alpha} \to_{\mathsf{q}} y$, whence $\mathsf{cl}(\mathcal{F})^{\alpha} \vee \mathsf{cl}(\mathcal{G})^{\alpha}$ q-converges to both $x$ and $y$, contradicting (H). Hence an $\alpha$-regular Hausdorff space is $\alpha$-Urysohn. □

**7.3. Complete Regularity.** A set $A$ is *completely within* $B$, $A \ll B$, if there is a continuous function $\varphi : (X, \mathsf{q}) \to [0, 1]$ (with the usual topology interpreted as a convergence space) such that $\varphi(A) \subseteq \{0\}$ and $\varphi(X \setminus B) \subseteq \{1\}$. By definition we have $\emptyset \ll A$ for all $X \neq \emptyset$ and $A \ll X$ for all $A \neq X$. Furthermore, $A' \subset A$, $B \subset B'$, and $A \ll B$ implies $F' \ll G$ and $F \ll G'$. To see this, we simply use the same function $\varphi$ that establishes $F \ll G$ an restrict $F$ or $X \setminus G$ to a subset.

**Lemma 12.** *$A \ll B$ implies $\mathsf{cl}(A) \subseteq B$.*

*Proof.* The lemma is trivial for $A = \emptyset$ or $B = X$. Hence we may assume that $A \neq \emptyset$ and $B \neq X$. In this case we have $\varphi(A) = \{0\}$ and $\varphi(X \setminus B) = \{1\}$. The function $\varphi$ is continuous, i.e., $\mathcal{F} \to_{\mathsf{q}} x$ implies $\varphi(F) \to \varphi(x)$. More explicitly, this means that, for all $\epsilon > 0$, there is a set $F \in \mathcal{F}$ such that $\varphi(F) \subseteq B_{\epsilon}(\varphi(x))$, where $B_{\epsilon}(y) = [0, 1] \cap (y - \epsilon, y + \epsilon)$. Now suppose $x \in \mathsf{cl}(A)$. Then $F \cap A \neq \emptyset$ for all $F\mathcal{F}$ and hence $\varphi(A) = \{0\}$ implies $0 \in \varphi(F) \subseteq B_{\epsilon}(\varphi(x))$, i.e., $\varphi(x) < \epsilon$. Thus $\varphi(x) \neq 1$, which implies $x \notin (X \setminus B)$ and hence $x \in B$. □

For each filter $\mathcal{F}$ we define

$$\mathcal{F}^\star = \{G \subseteq X | \exists F \in \mathcal{F} : F \ll G\}\uparrow \tag{28}$$

Lemma 12 implies $\mathcal{F}^* \subseteq \mathsf{cl}(\mathcal{F})$ since $\mathsf{cl}(F) \subseteq G$ whenever $F \ll G$.

A convergence space is

(CR) *completely regular* if $\mathcal{F} \to_{\mathsf{q}} x$ implies $\mathcal{F}^\star \to_{\mathsf{q}} x$.
(T3$\frac{1}{2}$) if it is complete regular and (T0).

The discussion above shows that a completely regular convergence space is regular, and hence the separation axiom (T3$\frac{1}{2}$) implies (T3).

**7.4. Normality.** Normal convergence spaces are introduced in [26] in the following way.

For $A \neq \emptyset$ we write

$$\mathcal{N}(A) = \{N | A \subseteq \mathsf{int}(N)\} = \bigcap_{x \in A} \mathcal{N}(x) \tag{29}$$

**Definition 14.** (QN) $(X, \mathsf{q})$ *is* quasi-normal *if for all pairs of non-empty disjoint closed sets A and B (i.e., $A = \mathsf{cl}(A) \neq \emptyset$, $B = \mathsf{cl}(B) \neq \emptyset$, and $A \cap B = \emptyset$) holds $\mathcal{N}(A) \vee \mathcal{N}(B) = \emptyset$.*

(N) $(X, \mathsf{q})$ *is* normal *if for all non-empty closed sets $A = \mathsf{cl}(A) \neq \emptyset$ holds $\mathcal{N}(A) \subseteq \mathsf{cl}(\mathcal{N}(A))$.*

(T4) $(X, \mathsf{q})$ *is* (T4) *if it is* (T1) *and normal.*

With these definitions [26] shows:

(QN) implies (N). In topological spaces (QN) and (N) are equivalent. (T4) implies regularity and hence (T3).

## 8. Compactness

**8.1. Adherence of a Filter.** The discussion below is extracted from [8].

**Definition 15.** *$x \in X$ is* adherent *to the filter $\mathcal{F}$ if there is $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G} \to_{\mathsf{q}} x$. The* adherence adh$\mathcal{F}$ *is the set of all points adherent to $\mathcal{F}$.*

In particular, therefore $\lim \mathcal{F} \subseteq \mathrm{adh}\mathcal{F}$. More generally we have

$$\mathrm{adh}\mathcal{F} = \left\{x \in X \middle| \exists \mathcal{G} \subseteq \mathcal{F} : \mathcal{G} \to_{\mathsf{q}} x\right\} == \cup_{\mathcal{G} \subseteq \mathcal{F}} \lim \mathcal{G}\,. \tag{30}$$

We remark two useful results from Fischer's [8] paper:
$\mathsf{cl}(A) = \mathrm{adh}\dot{A}$ for all $A \neq \emptyset$.
If $(X, \mathsf{q})$ is (T2) and $\mathcal{F}$ is a convergent filter, $\mathcal{F} \to_{\mathsf{q}} x$, then $\mathrm{adh}\mathcal{F} = \{x\}$.

**8.2. Compact Convergence Spaces.**

**Definition 16.** *A convergence space $(X, \mathsf{q})$ is* compact *if $\mathrm{adh}\mathcal{F} \neq \emptyset$ for every filter $\mathcal{F}$ on $X$.*

An equivalent formulation is

**Lemma 13.** *A convergence space* $(X, \mathsf{q})$ *is compact if and only if every ultrafilter on* $X$ *converges.*

**Lemma 14.** *Let* $(X, \mathsf{q})$ *be a compact (T2) pretopological space. Then* $\mathcal{F}$ *converges if and only if* $\mathrm{adh}\mathcal{F}$ *consists of a single point.*

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