Hypergraph Products

Lydia Gringmann

Bioinformatik Leipzig

October 23, 2009

Outline

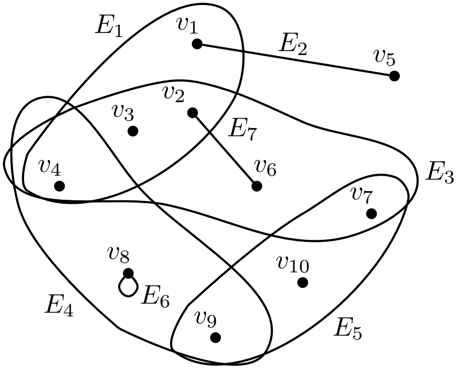
1. Hypergraphs
2. Hypergraph Products
3. Prime Factorization of the Cartesian Product

Hypergraphs

- A hypergraph is a pair $H = (V, \mathscr{E})$ with vertex set $V \neq \emptyset$ and a family of edges $\mathscr{E}$ where the edges are subsets of V.
- A hypergraph H is simple, if no edge of H is contained in any other edge.

Hypergraphs

Hypergraph $H = (V, \mathscr{E})$: $V = \{v_1, \ldots, v_{10}\}$, $\mathscr{E} = (E_1, \ldots, E_6)$



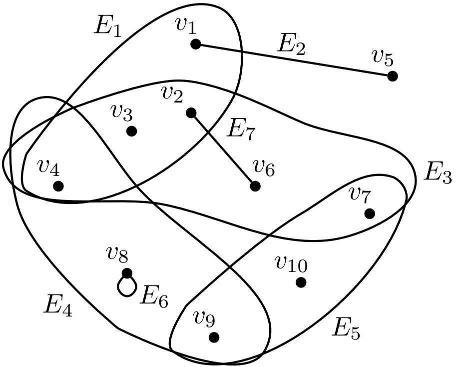
$$E_1 = \{v_1, v_2, v_3, v_4\}$$
$$E_2 = \{v_1, v_5\}$$
$$E_3 = \{v_2, v_3, v_4, v_6, v_7\}$$
$$E_4 = \{v_4, v_8, v_9\}$$
$$E_5 = \{v_7, v_9, v_{10}\}$$
$$E_6 = \{v_8\}$$
$$E_7 = \{v_2, v_6\}$$

Hypergraphs

- A hypergraph is a pair $H = (V, \mathscr{E})$ with vertex set $V \neq \emptyset$ and a family of edges $\mathscr{E}$ where the edges are subsets of V.
- A hypergraph H is simple, if no edge of H is contained in any other edge.

Hypergraphs

Hypergraph $H = (V, \mathscr{E})$: $V = \{v_1, \ldots, v_{10}\}$, $\mathscr{E} = (E_1, \ldots, E_6)$

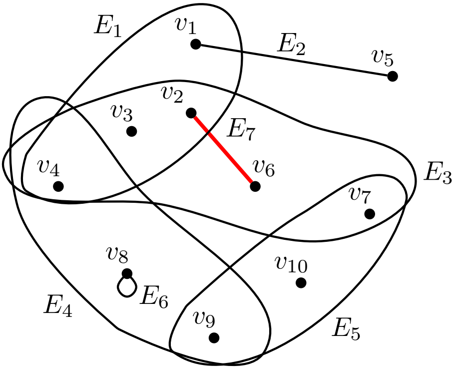


$$E_1 = \{v_1, v_2, v_3, v_4\}$$
$$E_2 = \{v_1, v_5\}$$
$$E_3 = \{v_2, v_3, v_4, v_6, v_7\}$$
$$E_4 = \{v_4, v_8, v_9\}$$
$$E_5 = \{v_7, v_9, v_{10}\}$$
$$E_6 = \{v_8\}$$
$$E_7 = \{v_2, v_6\}$$

H is not simple:

Hypergraphs

Hypergraph $H = (V, \mathscr{E})$: $V = \{v_1, \ldots, v_{10}\}$, $\mathscr{E} = (E_1, \ldots, E_6)$



$E_1 = \{v_1, v_2, v_3, v_4\}$

$E_2 = \{v_1, v_5\}$

$E_3 = \{v_2, v_3, v_4, v_6, v_7\}$

$E_4 = \{v_4, v_8, v_9\}$

$E_5 = \{v_7, v_9, v_{10}\}$

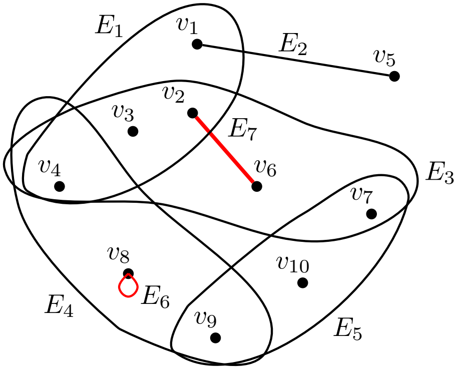
$E_6 = \{v_8\}$

$E_7 = \{v_2, v_6\}$

H is not simple: $E_7 \subseteq E_3$

Hypergraphs

Hypergraph $H = (V, \mathscr{E})$: $V = \{v_1, \ldots, v_{10}\}$, $\mathscr{E} = (E_1, \ldots, E_6)$

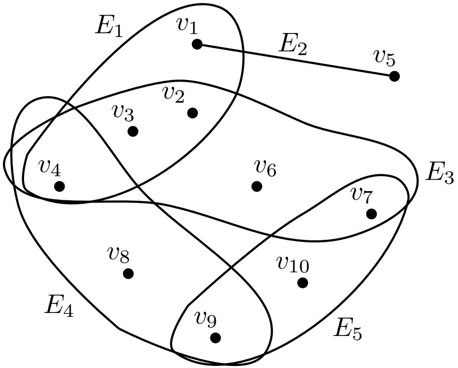


$$E_1 = \{v_1, v_2, v_3, v_4\}$$
$$E_2 = \{v_1, v_5\}$$
$$E_3 = \{v_2, v_3, v_4, v_6, v_7\}$$
$$E_4 = \{v_4, v_8, v_9\}$$
$$E_5 = \{v_7, v_9, v_{10}\}$$
$$E_6 = \{v_8\}$$
$$E_7 = \{v_2, v_6\}$$

H is not simple: $E_7 \subseteq E_3$, $E_6 \subseteq E_4$

Hypergraphs

Hypergraph $H = (V, \mathscr{E})$: $V = \{v_1, \ldots, v_{10}\}$, $\mathscr{E} = (E_1, \ldots, E_5)$



$E_1 = \{v_1, v_2, v_3, v_4\}$

$E_2 = \{v_1, v_5\}$

$E_3 = \{v_2, v_3, v_4, v_6, v_7\}$

$E_4 = \{v_4, v_8, v_9\}$

$E_5 = \{v_7, v_9, v_{10}\}$

H is simple

Hypergraph Products

- For all products $H_1 \star H_1$ defined in this section:

$$V(H_1 \star H_2) = V(H_1) \times V(H_2)$$

1. Restriction of the products on graphs are the common graph products
2. Assiociativity
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization

Hypergraph Products

- For all products $H_1 \star H_1$ defined in this section:

$$V(H_1 \star H_2) = V(H_1) \times V(H_2)$$

Wish to have:

1. Restriction of the products on graphs are the common graph products
2. Assiociativity
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization

Hypergraph Products

- For all products $H_1 \star H_1$ defined in this section:

$$V(H_1 \star H_2) = V(H_1) \times V(H_2)$$

Wish to have:

1. Restriction of the products on graphs are the common graph products
2. Assiociativity
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization

Hypergraph Products

- For all products $H_1 \star H_1$ defined in this section:

$$V(H_1 \star H_2) = V(H_1) \times V(H_2)$$

Wish to have:

1. Restriction of the products on graphs are the common graph products
2. Assiociativity
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization

Hypergraph Products

- For all products $H_1 \star H_1$ defined in this section:

$$V(H_1 \star H_2) = V(H_1) \times V(H_2)$$

Wish to have:

1. Restriction of the products on graphs are the common graph products
2. Assiociativity
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization

Hypergraph Products

- For all products $H_1 \star H_1$ defined in this section:

$$V(H_1 \star H_2) = V(H_1) \times V(H_2)$$

Wish to have:

1. Restriction of the products on graphs are the common graph products
2. Assiociativity
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization

Hypergraph Products

- For all products $H_1 \star H_1$ defined in this section:

$$V(H_1 \star H_2) = V(H_1) \times V(H_2)$$

Wish to have:

1. Restriction of the products on graphs are the common graph products
2. Assiociativity
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization

Hypergraph Products

- For all products $H_1 \star H_1$ defined in this section:

$$V(H_1 \star H_2) = V(H_1) \times V(H_2)$$

Wish to have:

1. Restriction of the products on graphs are the common graph products
2. Assiociativity
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization (under some constraints)

Cartesian Product

Edge set of the Cartesian Product $H = H_1 \Box H_2$ of two hypergraphs H_1, H_2

$$\begin{aligned}\mathscr{E}(H) = &\big\{\{x\} \times F : x \in V(H_1), F \in \mathscr{E}(H_2)\big\} \\ &\cup \big\{E \times \{y\} : E \in \mathscr{E}(H_1), y \in V(H_2)\big\}\end{aligned}$$

Cartesian Product

Edge set of the Cartesian Product $H = H_1 \Box H_2$ of two hypergraphs H_1, H_2

$$\begin{aligned}\mathscr{E}(H) = \big\{\{x\} \times F : x \in V(H_1), F \in \mathscr{E}(H_2)\big\} \\ \cup \big\{E \times \{y\} : E \in \mathscr{E}(H_1), y \in V(H_2)\big\}\end{aligned}$$

I.e., $\{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathscr{E}(H_1 \Box H_2)$

Cartesian Product

Edge set of the Cartesian Product $H = H_1 \Box H_2$ of two hypergraphs H_1, H_2

$$\mathscr{E}(H) = \big\{\{x\} \times F : x \in V(H_1), F \in \mathscr{E}(H_2)\big\} \\ \cup \big\{E \times \{y\} : E \in \mathscr{E}(H_1), y \in V(H_2)\big\}$$

I.e., $\{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathscr{E}(H_1 \Box H_2) \iff$

(i) $\{x_1, \ldots, x_r\} \in \mathscr{E}(H_1)$ and $y_1 = \ldots = y_r$,or

Cartesian Product

Edge set of the Cartesian Product $H = H_1 \Box H_2$ of two hypergraphs H_1, H_2

$$\begin{aligned}\mathscr{E}(H) = \big\{\{x\} \times F : x \in V(H_1), F \in \mathscr{E}(H_2)\big\} \\ \cup \big\{E \times \{y\} : E \in \mathscr{E}(H_1), y \in V(H_2)\big\}\end{aligned}$$

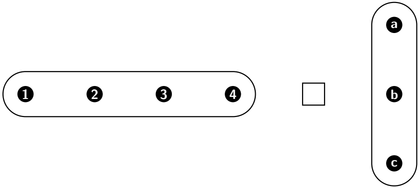
I.e., $\{(x_1, y_1), \dots, (x_r, y_r)\} \in \mathscr{E}(H_1 \Box H_2) \quad \Longleftrightarrow$

(i) $\{x_1, \dots, x_r\} \in \mathscr{E}(H_1)$ and $y_1 = \dots = y_r$,or

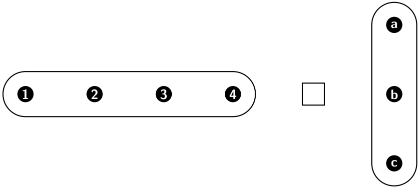
(ii) $\{y_1, \dots, y_r\} \in \mathscr{E}(H_2)$ and $x_1 = \dots = x_r$

Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3,4\}, \mathscr{E}_1 = (\{1,2,3,4\})$

Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3,4\}, \mathscr{E}_1 = (\{1,2,3,4\})$

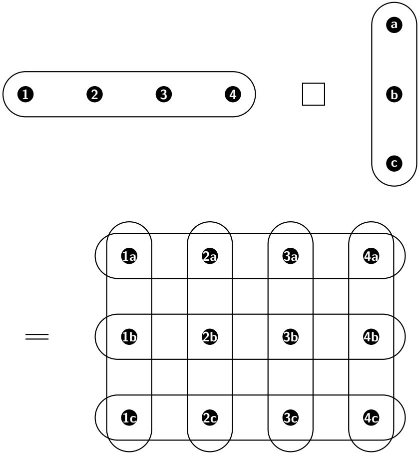


Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3,4\}, \mathscr{E}_1 = (\{1,2,3,4\})$ and $H_2 = (V_2, \mathscr{E}_2)$ with $V_2 = \{a,b,c\}, \mathscr{E}_2 = (\{a,b,c\})$



$=$

Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3,4\}, \mathscr{E}_1 = (\{1,2,3,4\})$ and $H_2 = (V_2, \mathscr{E}_2)$ with $V_2 = \{a,b,c\}, \mathscr{E}_2 = (\{a,b,c\})$



Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3,4\}, \mathscr{E}_1 = (\{1,2,3,4\})$ and $H_2 = (V_2, \mathscr{E}_2)$ with $V_2 = \{a,b,c\}, \mathscr{E}_2 = (\{a,b,c\})$ and their Cartesian product $H_1 \Box H_2$

Direct Product 1 (preserving minimal rank)

$\{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathscr{E}(H_1 \times_1 H_2) \iff$

Direct Product 1 (preserving minimal rank)

$\{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathscr{E}(H_1 \times_1 H_2) \iff$

(i) $\{x_1, \ldots, x_r\} \in \mathscr{E}(H_1)$ and $\{y_1, \ldots, y_r\} \subset E^2 \in \mathscr{E}(H_2)$, or

Direct Product 1 (preserving minimal rank)

$\{(x_1,y_1),\ldots,(x_r,y_r)\} \in \mathscr{E}(H_1 \times_1 H_2) \quad \iff$

(i) $\{x_1,\ldots,x_r\} \in \mathscr{E}(H_1)$ and $\{y_1,\ldots,y_r\} \subset E^2 \in \mathscr{E}(H_2)$, or

(ii) $\{y_1,\ldots,y_r\} \in \mathscr{E}(H_2)$ and $\{x_1,\ldots,x_r\} \subset E^1 \in \mathscr{E}(H_1)$

Direct Product 1 (preserving minimal rank)

$\{(x_1,y_1),\ldots,(x_r,y_r)\} \in \mathscr{E}(H_1 \times_1 H_2) \quad \Longleftrightarrow$

(i) $\{x_1,\ldots,x_r\} \in \mathscr{E}(H_1)$ and $\{y_1,\ldots,y_r\} \subset E^2 \in \mathscr{E}(H_2)$, or

(ii) $\{y_1,\ldots,y_r\} \in \mathscr{E}(H_2)$ and $\{x_1,\ldots,x_r\} \subset E^1 \in \mathscr{E}(H_1)$

More formal:

Direct Product 1 (preserving minimal rank)

$\{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathscr{E}(H_1 \times_1 H_2) \iff$

(i) $\{x_1, \ldots, x_r\} \in \mathscr{E}(H_1)$ and $\{y_1, \ldots, y_r\} \subset E^2 \in \mathscr{E}(H_2)$, or

(ii) $\{y_1, \ldots, y_r\} \in \mathscr{E}(H_2)$ and $\{x_1, \ldots, x_r\} \subset E^1 \in \mathscr{E}(H_1)$

More formal:
Edge set of the Direct Product 1 $H = H_1 \times_1 H_2$ of two hypergraphs H_1, H_2:

Direct Product 1 (preserving minimal rank)

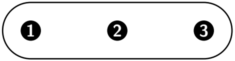
$\{(x_1,y_1),\dots,(x_r,y_r)\} \in \mathscr{E}(H_1 \times_1 H_2) \quad \Longleftrightarrow$

(i) $\{x_1,\dots,x_r\} \in \mathscr{E}(H_1)$ and $\{y_1,\dots,y_r\} \subset E^2 \in \mathscr{E}(H_2)$, or

(ii) $\{y_1,\dots,y_r\} \in \mathscr{E}(H_2)$ and $\{x_1,\dots,x_r\} \subset E^1 \in \mathscr{E}(H_1)$

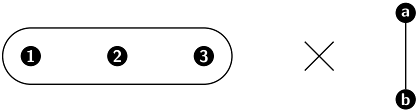
More formal:
Edge set of the Direct Product 1 $H = H_1 \times_1 H_2$ of two hypergraphs H_1, H_2:

$$\mathscr{E}(H_1 \times_1 H_2) := \Big\{ \bigcup_{x\in E} \{(x,\pi(x))\} : \pi : E \to F \text{ injective}, E \in \mathscr{E}_1, F \in \mathscr{E}_2 \Big\} \cup \Big\{ \bigcup_{y\in F} \{(\pi'(y),y)\} : \pi' : F \to E \text{ injective}, E \in \mathscr{E}_1, F \in \mathscr{E}_2 \Big\}$$

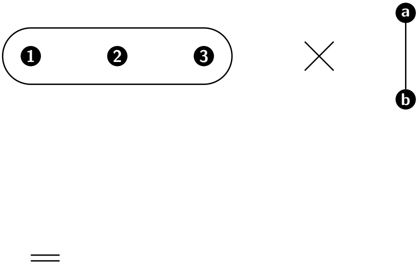


Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$

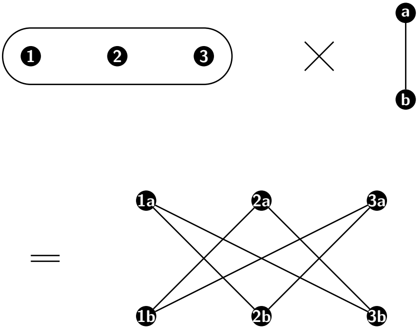
Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$



Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$ and $H_2 = (V_2, \mathscr{E}_2)$ with $V_2 = \{a,b\}, \mathscr{E}_2 = (\{a,b\})$



Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$ and $H_2 = (V_2, \mathscr{E}_2)$ with $V_2 = \{a,b\}, \mathscr{E}_2 = (\{a,b\})$



Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$ and $H_2 = (V_2, \mathscr{E}_2)$ with $V_2 = \{a,b\}, \mathscr{E}_2 = (\{a,b\})$ and their direct product 1 $H_1 \times_1 H_2$

Direct Product 2 (preserving maximal rank)

$\{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathscr{E}(H_1 \times_2 H_2) \qquad \Longleftrightarrow$

Direct Product 2 (preserving maximal rank)

$\{(x_1,y_1),\ldots,(x_r,y_r)\} \in \mathscr{E}(H_1 \times_2 H_2) \qquad \iff$

(i) $\{x_1,\ldots,x_r\} \in \mathscr{E}(H_1)$ and $\exists E^2 \in \mathscr{E}(H_2)$ s.t. $\{y_1,\ldots y_r\}$ is a family of elements of E^2, and $E^2 \subseteq \{y_1,\ldots,y_r\}$, or

Direct Product 2 (preserving maximal rank)

$\{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathscr{E}(H_1 \times_2 H_2) \quad \Longleftrightarrow$

(i) $\{x_1, \ldots, x_r\} \in \mathscr{E}(H_1)$ and $\exists E^2 \in \mathscr{E}(H_2)$ s.t. $\{y_1, \ldots y_r\}$ is a family of elements of E^2, and $E^2 \subseteq \{y_1, \ldots, y_r\}$, or

(ii) $\{y_1, \ldots, y_r\} \in \mathscr{E}(H_2)$ and $\exists E^1 \in \mathscr{E}(H_1)$ s. t. $\{x_1, \ldots x_r\}$ is a family of elements of E^1, and $E^1 \subseteq \{x_1, \ldots, x_r\}$.

Direct Product 2 (preserving maximal rank)

$\{(x_1,y_1),\dots,(x_r,y_r)\} \in \mathscr{E}(H_1 \times_2 H_2) \qquad \Longleftrightarrow$

(i) $\{x_1,\dots,x_r\} \in \mathscr{E}(H_1)$ and $\exists E^2 \in \mathscr{E}(H_2)$ s.t. $\{y_1,\dots y_r\}$ is a family of elements of E^2, and $E^2 \subseteq \{y_1,\dots,y_r\}$, or

(ii) $\{y_1,\dots,y_r\} \in \mathscr{E}(H_2)$ and $\exists E^1 \in \mathscr{E}(H_1)$ s. t. $\{x_1,\dots x_r\}$ is a family of elements of E^1, and $E^1 \subseteq \{x_1,\dots,x_r\}$.

More formal:

Direct Product 2 (preserving maximal rank)

$\{(x_1,y_1),\ldots,(x_r,y_r)\} \in \mathscr{E}(H_1 \times_2 H_2) \qquad \Longleftrightarrow$

(i) $\{x_1,\ldots,x_r\} \in \mathscr{E}(H_1)$ and $\exists E^2 \in \mathscr{E}(H_2)$ s.t. $\{y_1,\ldots y_r\}$ is a family of elements of E^2, and $E^2 \subseteq \{y_1,\ldots,y_r\}$, or

(ii) $\{y_1,\ldots,y_r\} \in \mathscr{E}(H_2)$ and $\exists E^1 \in \mathscr{E}(H_1)$ s. t. $\{x_1,\ldots x_r\}$ is a family of elements of E^1, and $E^1 \subseteq \{x_1,\ldots,x_r\}$.

More formal:
Edge set of the Direct Product 2 $H = H_1 \times_2 H_2$ of two hypergraphs H_1, H_2:

Direct Product 2 (preserving maximal rank)

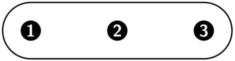
$\{(x_1,y_1),\ldots,(x_r,y_r)\} \in \mathscr{E}(H_1 \times_2 H_2) \quad \Longleftrightarrow$

(i) $\{x_1,\ldots,x_r\} \in \mathscr{E}(H_1)$ and $\exists E^2 \in \mathscr{E}(H_2)$ s.t. $\{y_1,\ldots y_r\}$ is a family of elements of E^2, and $E^2 \subseteq \{y_1,\ldots,y_r\}$, or

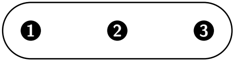
(ii) $\{y_1,\ldots,y_r\} \in \mathscr{E}(H_2)$ and $\exists E^1 \in \mathscr{E}(H_1)$ s. t. $\{x_1,\ldots x_r\}$ is a family of elements of E^1, and $E^1 \subseteq \{x_1,\ldots,x_r\}$.

More formal:
Edge set of the Direct Product 2 $H = H_1 \times_2 H_2$ of two hypergraphs H_1, H_2:

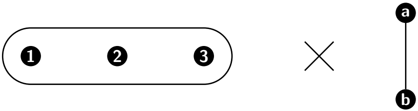
$$\mathscr{E}(H_1 \times H_2) := \{ \bigcup_{x\in E} \{(x,\varphi(x))\} \mid \varphi : E \to F \text{ surjective}, E \in \mathscr{E}_1, F \in \mathscr{E}_2\} \cup \{ \bigcup_{y\in F} \{(\varphi'(y),y)\} : \varphi' : F \to E \text{ surjective}, E \in \mathscr{E}_1, F \in \mathscr{E}_2\}$$



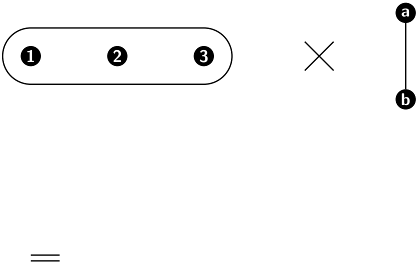
Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$



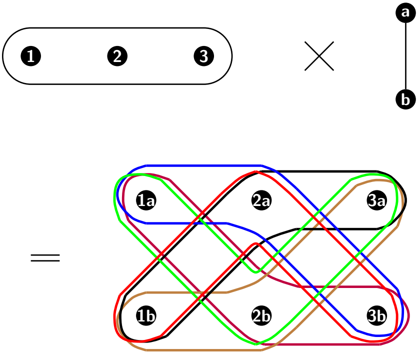
Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$



Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$ and $H_2 = (V_2, \mathscr{E}_2)$ with $V_2 = \{a,b\}, \mathscr{E}_2 = (\{a,b\})$



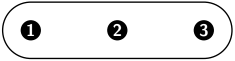
Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$ and $H_2 = (V_2, \mathscr{E}_2)$ with $V_2 = \{a,b\}, \mathscr{E}_2 = (\{a,b\})$



Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$ and $H_2 = (V_2, \mathscr{E}_2)$ with $V_2 = \{a,b\}, \mathscr{E}_2 = (\{a,b\})$ and their Direct product 2 $H_1 \times_2 H_2$

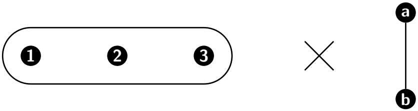
Direct Product 3

Let $E \subseteq V(H_1 \times H_2)$. $E \in \mathscr{E}(H_1 \times_3 H_2) \iff$
It has the form $E = \{(x,y)\} \cup \big((E^1 \setminus \{x\}) \times E^2 \setminus \{y\}\big)$
for $x \in E^1 \in \mathscr{E}(H_1), y \in E^2 \in \mathscr{E}(H_2)$

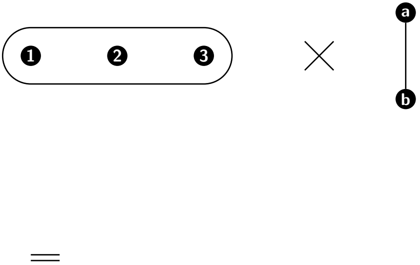


Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$

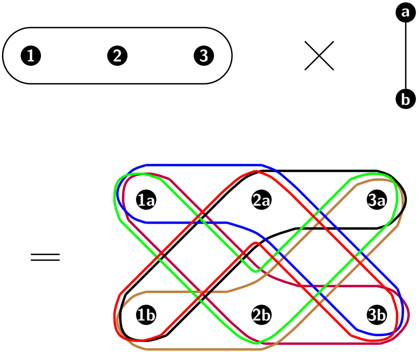
Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$



Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$ and $H_2 = (V_2, \mathscr{E}_2)$ with $V_2 = \{a,b\}, \mathscr{E}_2 = (\{a,b\})$



Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$ and $H_2 = (V_2, \mathscr{E}_2)$ with $V_2 = \{a,b\}, \mathscr{E}_2 = (\{a,b\})$



Two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ with $V_1 = \{1,2,3\}, \mathscr{E}_1 = (\{1,2,3\})$ and $H_2 = (V_2, \mathscr{E}_2)$ with $V_2 = \{a,b\}, \mathscr{E}_2 = (\{a,b\})$ and their Direct product 3 $H_1 \times_3 H_2$ (In this case: $H_1 \times_3 H_2 = H_1 \times_2 H_2$)

More Examples

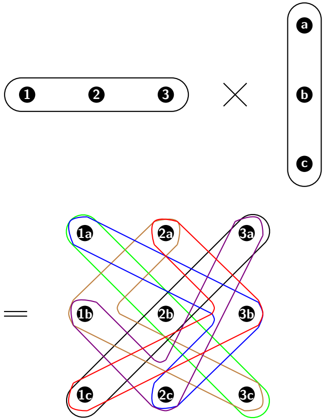


Figure: $H_1 \times_1 H_2 = H_1 \times_2 H_2$; $H_1 = (\{1,2,3\},(\{1,2,3\}))$, $H_2 = (\{a,b,c\},(\{a,b,c\}))$

More Examples

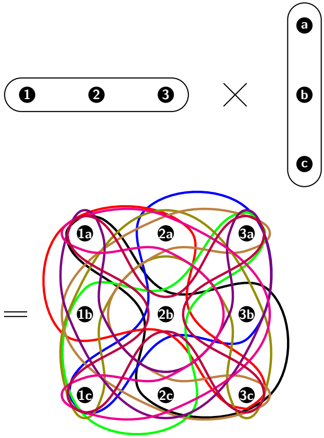
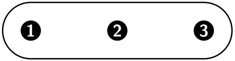


Figure: $H_1 \times_3 H_2$; $H_1 = (\{1,2,3\},(\{1,2,3\}))$, $H_2 = (\{a,b,c\},(\{a,b,c\}))$

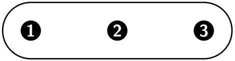
Strong Product 1

Edge set of the strong products $=$ Edge set of the direct products $\cup$ Edge set of the Cartesian product:



Strong Product 1

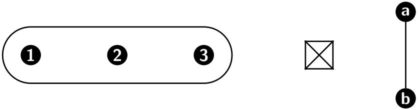
Edge set of the strong products $=$ Edge set of the direct products $\cup$ Edge set of the Cartesian product:



$\boxtimes$

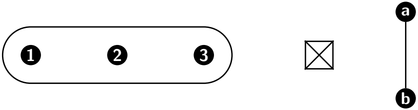
Strong Product 1

Edge set of the strong products = Edge set of the direct products ∪ Edge set of the Cartesian product:



Strong Product 1

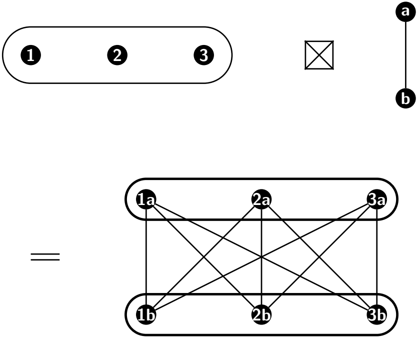
Edge set of the strong products $=$ Edge set of the direct products $\cup$ Edge set of the Cartesian product:



$=$

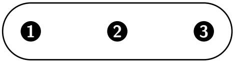
Strong Product 1

Edge set of the strong products = Edge set of the direct products ∪ Edge set of the Cartesian product:



Strong Products 2/3

Edge set of the strong products $=$ Edge set of the direct products $\cup$ Edge set of the Cartesian product:

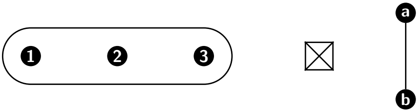


Strong Products 2/3

Edge set of the strong products $=$ Edge set of the direct products $\cup$ Edge set of the Cartesian product:

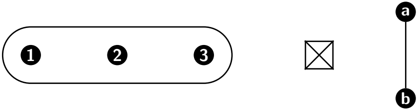
Strong Products 2/3

Edge set of the strong products = Edge set of the direct products ∪ Edge set of the Cartesian product:



Strong Products 2/3

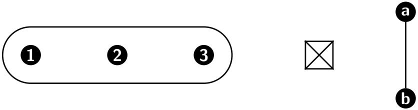
Edge set of the strong products = Edge set of the direct products ∪ Edge set of the Cartesian product:



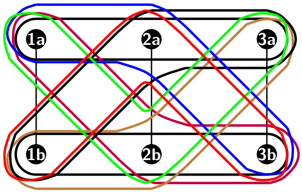
=

Strong Products 2/3

Edge set of the strong products = Edge set of the direct products ∪ Edge set of the Cartesian product:



=



Hypergraph Products

We have:

1. Restriction of the products on graphs are the common graph products
2. Assiociativity
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization (under some constraints)

Hypergraph Products

We have:

1. Restriction of the products on graphs are the common graph products $\surd$
2. Assiociativity
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization (under some constraints)

Hypergraph Products

We have:

1. Restriction of the products on graphs are the common graph products $\surd$
2. Assiociativity $\surd$
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization (under some constraints)

Hypergraph Products

We have:

1. Restriction of the products on graphs are the common graph products $\surd$
2. Assiociativity $\surd$
3. Commutativity $\surd$
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization (under some constraints)

Hypergraph Products

We have:

1. Restriction of the products on graphs are the common graph products $\surd$
2. Assiociativity $\surd$
3. Commutativity $\surd$
4. Distributivity w.r.t. the disjoint union $\surd$
5. Products of simple hypergraphs are simple
6. Unique prime factorization (under some constraints)

Hypergraph Products

We have:

1. Restriction of the products on graphs are the common graph products $\surd$
2. Assiociativity $\surd$
3. Commutativity $\surd$
4. Distributivity w.r.t. the disjoint union $\surd$
5. Products of simple hypergraphs are simple $\surd$
6. Unique prime factorization (under some constraints)

Hypergraph Products

We have:

1. Restriction of the products on graphs are the common graph products $\surd$
2. Assiociativity $\surd$
3. Commutativity $\surd$
4. Distributivity w.r.t. the disjoint union $\surd$
5. Products of simple hypergraphs are simple $\surd$
6. Unique prime factorization (under some constraints) **?**

Grid Property

- 2 incident edges E, F of a Cartesian product belonging to two different factors span exactly one $|E| \times |F|$-grid

Grid Property

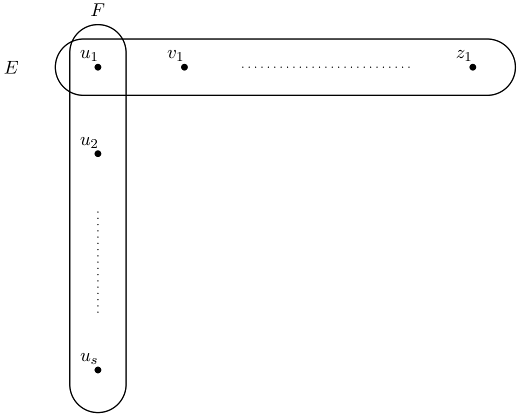
- 2 incident edges E, F of a Cartesian product belonging to two different factors span exactly one $|E| \times |F|$-grid

$\longrightarrow$ grid-property

Grid Property

- 2 incident edges E, F of a Cartesian product belonging to two different factors span exactly one $|E| \times |F|$-grid

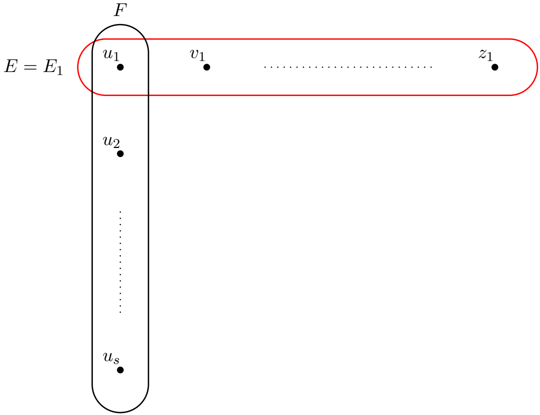
$\longrightarrow$ grid-property



Grid Property

- 2 incident edges E, F of a Cartesian product belonging to two different factors span exactly one $|E| \times |F|$-grid

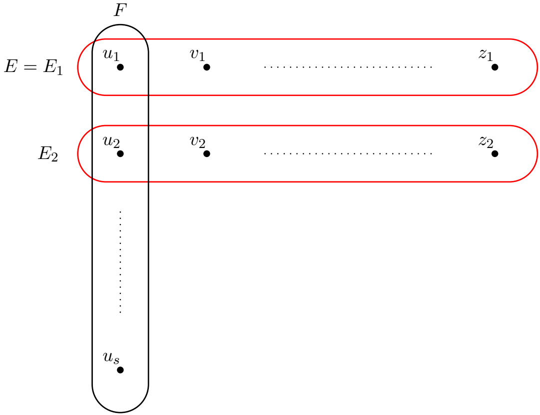
$\longrightarrow$ grid-property



Grid Property

- 2 incident edges E, F of a Cartesian product belonging to two different factors span exactly one $|E| \times |F|$-grid

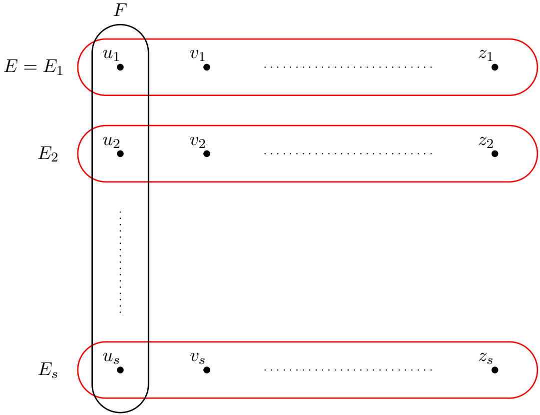
$\longrightarrow$ grid-property



Grid Property

- 2 incident edges E, F of a Cartesian product belonging to two different factors span exactly one $|E| \times |F|$-grid

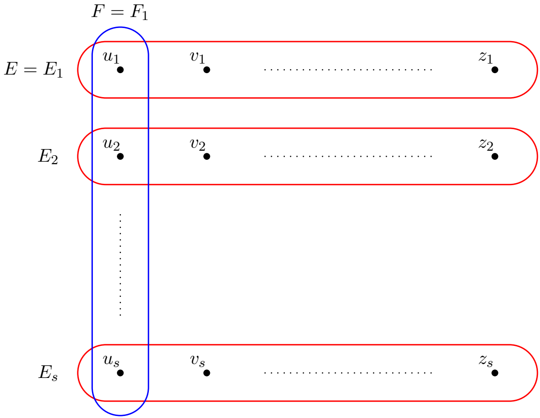
$\longrightarrow$ grid-property



Grid Property

- 2 incident edges E, F of a Cartesian product belonging to two different factors span exactly one $|E| \times |F|$-grid

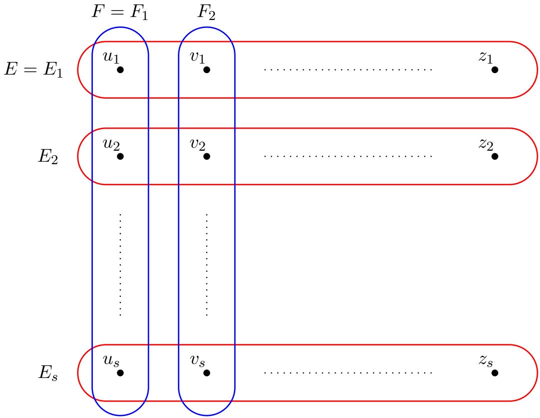
$\longrightarrow$ grid-property



Grid Property

- 2 incident edges E, F of a Cartesian product belonging to two different factors span exactly one $|E| \times |F|$-grid

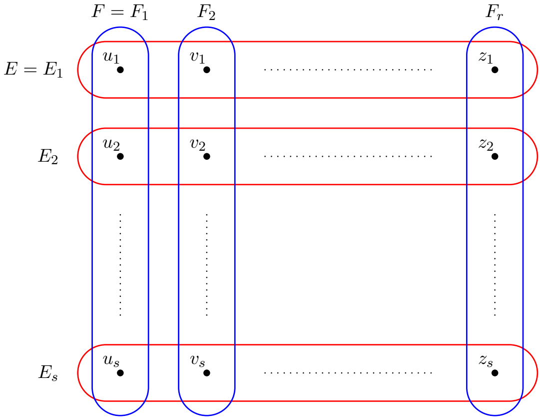
$\longrightarrow$ grid-property



Grid Property

- 2 incident edges E, F of a Cartesian product belonging to two different factors span exactly one $|E| \times |F|$-grid

$\longrightarrow$ grid-property



"‘starting"’-relation δ on $\mathscr{E}(H)$:

- $E \delta F \iff$

(i) $|E \cap F| > 1$

(ii) E and F are opposite edges of a four-cycle

(iii) $|E \cap F| = 1$ and $\nexists (|E| \times |F|)$-grid without diagonals containing them.

- δ^* suffices the grid property

"‘starting"’-relation δ on $\mathscr{E}(H)$:

- $E \delta F \iff$

(i) $|E \cap F| > 1$

(ii) E and F are opposite edges of a four-cycle

(iii) $|E \cap F| = 1$ and $\nexists(|E| \times |F|)$-grid without diagonals containing them.

- δ^* suffices the grid property

"'starting"'-relation δ on $\mathscr{E}(H)$:

- $E \delta F \iff$

(i) $|E \cap F| > 1$

(ii) E and F are opposite edges of a four-cycle

(iii) $|E \cap F| = 1$ and $\nexists(|E| \times |F|)$-grid without diagonals containing them.

- δ^* suffices the grid property

"‘starting"’-relation δ on $\mathscr{E}(H)$:

- $E \delta F \iff$

(i) $|E \cap F| > 1$

(ii) E and F are opposite edges of a four-cycle

(iii) $|E \cap F| = 1$ and $\nexists (|E| \times |F|)$-grid without diagonals containing them.

- δ^* suffices the grid property

"'starting"'-relation δ on $\mathscr{E}(H)$:

- $E \delta F \iff$

(i) $|E \cap F| > 1$

(ii) E and F are opposite edges of a four-cycle

(iii) $|E \cap F| = 1$ and $\nexists (|E| \times |F|)$-grid without diagonals containing them.

- δ^* suffices the grid property

"‘starting"’-relation δ on $\mathscr{E}(H)$:

- $E\delta F \iff$

(i) $|E \cap F| > 1$

(ii) E and F are opposite edges of a four-cycle

(iii) $|E \cap F| = 1$ and $\nexists(|E| \times |F|)$-grid without diagonals containing them.

- δ^* suffices the grid property

- We have: relation δ^* with

$$E\delta^* F \Rightarrow E \text{ and } F \text{ belong to the same prime factor.}$$

- We want: relation σ with

$$E\sigma F \Leftrightarrow E \text{ and } F \text{ belong to the same prime factor.}$$

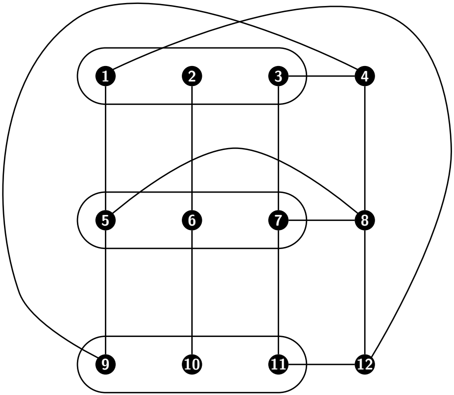
- We have: relation δ^* with

$$E\delta^* F \Rightarrow E \text{ and } F \text{ belong to the same prime factor.}$$

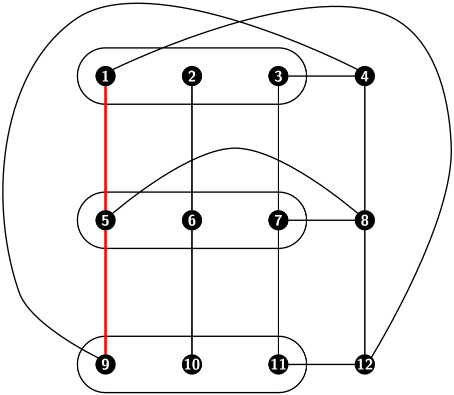
- We want: relation σ with

$$E\sigma F \Leftrightarrow E \text{ and } F \text{ belong to the same prime factor.}$$

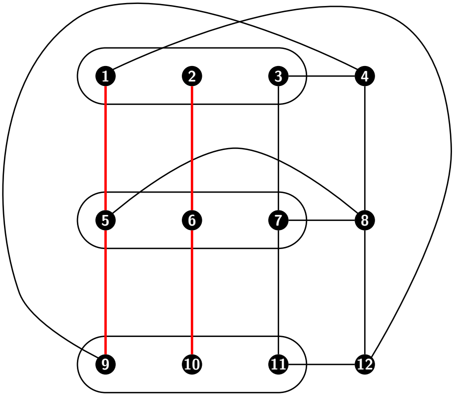
Equivalence classes of δ^*:



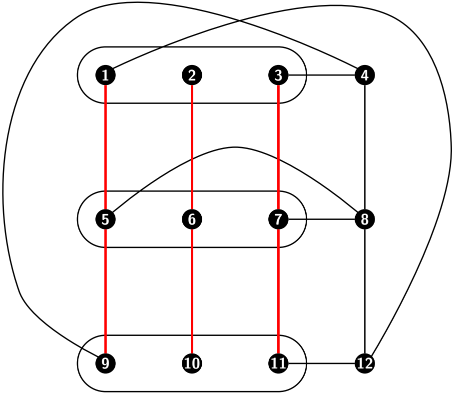
Equivalence classes of δ^*:



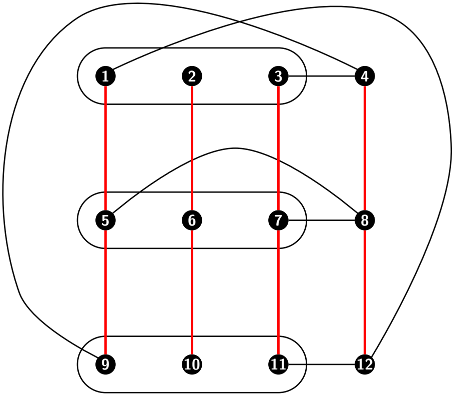
Equivalence classes of δ^*:



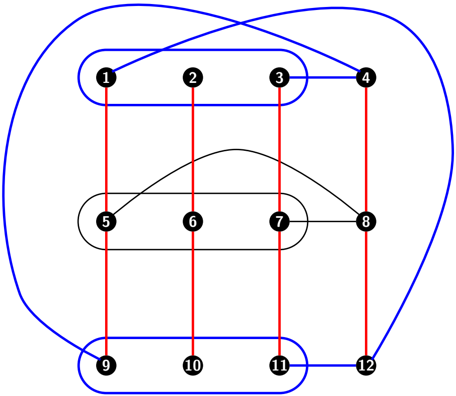
Equivalence classes of δ^*:



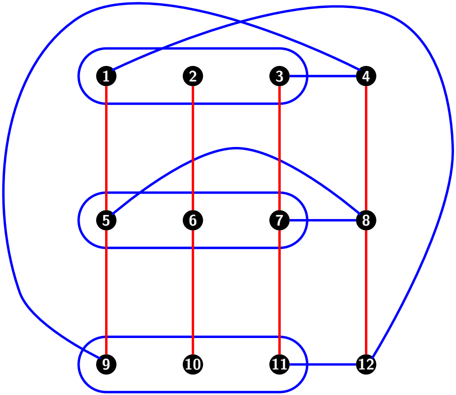
Equivalence classes of δ^*:



Equivalence classes of δ^*:

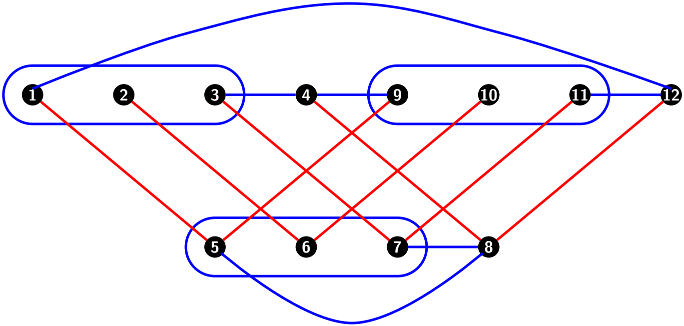


Equivalence classes of δ^*:



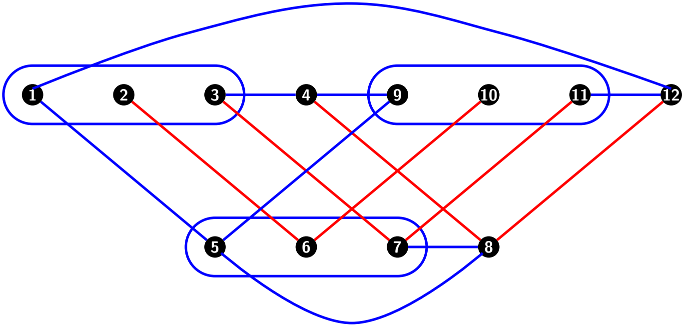
Convex hull of δ, $\mathscr{C}(\delta)$

i.e. the smallest convex equivalence relation containing δ:



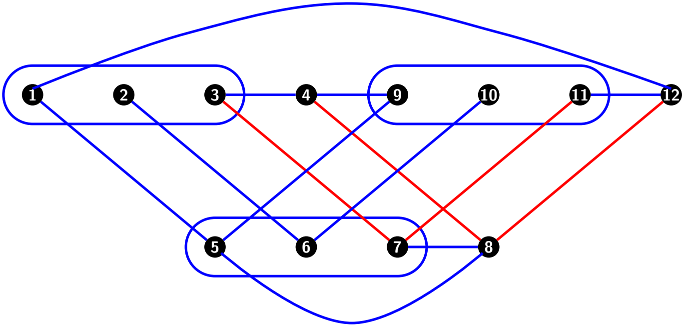
Convex hull of δ, $\mathscr{C}(\delta)$

i.e. the smallest convex equivalence relation containing δ:



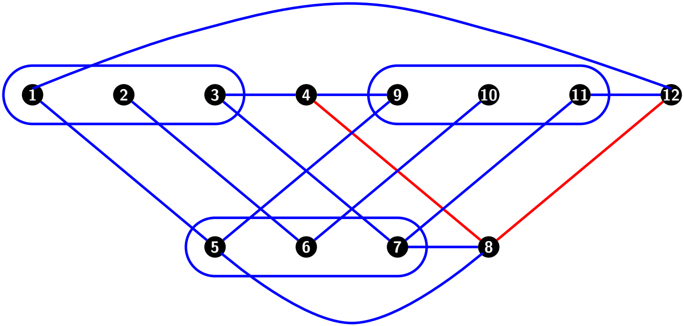
Convex hull of δ, $\mathscr{C}(\delta)$

i.e. the smallest convex equivalence relation containing δ:



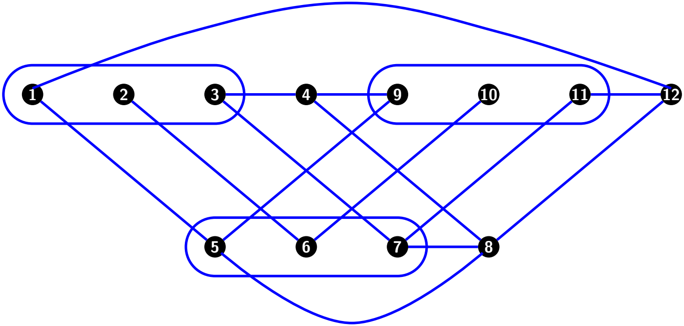
Convex hull of δ, $\mathscr{C}(\delta)$

i.e. the smallest convex equivalence relation containing δ:



Convex hull of δ, $\mathscr{C}(\delta)$

i.e. the smallest convex equivalence relation containing δ:



Theorem

Every connected Hypergraph has a unique prime factorization.

Theorem

The relation corresponding to the unique prime factorization of a connected hypergraph is the convex hull of the δ-relation, $\sigma = \mathscr{C}(\delta)$

Theorem

Every connected Hypergraph has a unique prime factorization.

Theorem

The relation corresponding to the unique prime factorization of a connected hypergraph is the convex hull of the δ*-relation,* $\sigma = \mathscr{C}(\delta)$

Hypergraph Products

We have for the Cartesian Product

1. Restriction of the products on graphs are the common graph products $\surd$
2. Assiociativity $\surd$
3. Commutativity $\surd$
4. Distributivity w.r.t. the disjoint union $\surd$
5. Products of simple hypergraphs are simple $\surd$
6. Unique prime factorization (for connected hypergraphs) **?**

Hypergraph Products

We have for the Cartesian Product

1. Restriction of the products on graphs are the common graph products $\surd$
2. Assiociativity $\surd$
3. Commutativity $\surd$
4. Distributivity w.r.t. the disjoint union $\surd$
5. Products of simple hypergraphs are simple $\surd$
6. Unique prime factorization (for connected hypergraphs) $\surd$

Thanks to Marc Hellmuth and Peter Stadler!

Thanks to Marc Hellmuth and Peter Stadler!

Thank you for your attention!