Hypergraph Products

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Outline

1. Hypergraphs
2. Hypergraph Products
3. Prime Factorization of the Cartesian Product
Hypergraphs

• A hypergraph is a pair $H = (V, \mathcal{E})$ with vertex set $V \neq \emptyset$ and a family of edges $\mathcal{E}$ where the edges are subsets of $V$.

• A hypergraph $H$ is simple, if no edge of $H$ is contained in any other edge.
Hypergraphs

Hypergraph $H = (V, \mathcal{E})$: $V = \{v_1, \ldots, v_{10}\}$, $\mathcal{E} = (E_1, \ldots, E_6)$

$E_1 = \{v_1, v_2, v_3, v_4\}$  
$E_2 = \{v_1, v_5\}$  
$E_3 = \{v_2, v_3, v_4, v_6, v_7\}$  
$E_4 = \{v_4, v_8, v_9\}$  
$E_5 = \{v_7, v_9, v_{10}\}$  
$E_6 = \{v_8\}$  
$E_7 = \{v_2, v_6\}$
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$E_5 = \{v_7, v_9, v_{10}\}$
$E_6 = \{v_8\}$
$E_7 = \{v_2, v_6\}$

$H$ is not simple:
Hypergraphs

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$H$ is not simple: $E_7 \subseteq E_3$
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$H$ is not simple: $E_7 \subseteq E_3$, $E_6 \subseteq E_4$
Hypergraphs

Hypergraph $H = (V, E)$: $V = \{v_1, \ldots, v_{10}\}$, $E = (E_1, \ldots, E_5)$

$E_1 = \{v_1, v_2, v_3, v_4\}$
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$H$ is simple
Hypergraph Products

- For all products $H_1 \star H_1$ defined in this section:

  \[ V(H_1 \star H_2) = V(H_1) \times V(H_2) \]

1. Restriction of the products on graphs are the common graph products
2. Associativity
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization
Hypergraph Products

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5. Products of simple hypergraphs are simple
6. Unique prime factorization (under some constraints)
Cartesian Product

Edge set of the Cartesian Product $H = H_1 \square H_2$ of two hypergraphs $H_1, H_2$

$$\mathcal{E}(H) = \left\{ \{x\} \times F : x \in V(H_1), F \in \mathcal{E}(H_2) \right\} \cup \left\{ E \times \{y\} : E \in \mathcal{E}(H_1), y \in V(H_2) \right\}$$
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$$\cup \left\{ E \times \{y\} : E \in \mathcal{E}(H_1), y \in V(H_2) \right\}$$

I.e., $\{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathcal{E}(H_1 \square H_2)$
Cartesian Product

Edge set of the Cartesian Product $H = H_1 \Box H_2$ of two hypergraphs $H_1, H_2$

$$E(H) = \{\{x\} \times F : x \in V(H_1), F \in E(H_2)\}$$
$$\cup \{E \times \{y\} : E \in E(H_1), y \in V(H_2)\}$$

I.e., $\{(x_1, y_1), \ldots, (x_r, y_r)\} \in E(H_1 \Box H_2) \iff$

(i) $\{x_1, \ldots, x_r\} \in E(H_1)$ and $y_1 = \ldots = y_r$, or
Hypergraphs

Hypergraph Products

Prime Factorization w.r.t the Cartesian Product

Cartesian Product

Edge set of the Cartesian Product $H = H_1 \square H_2$ of two hypergraphs $H_1, H_2$

$\mathcal{E}(H) = \{\{x\} \times F : x \in V(H_1), F \in \mathcal{E}(H_2)\} \cup \{E \times \{y\} : E \in \mathcal{E}(H_1), y \in V(H_2)\}$

I.e., $\{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathcal{E}(H_1 \square H_2) \iff$

(i) $\{x_1, \ldots, x_r\} \in \mathcal{E}(H_1)$ and $y_1 = \ldots = y_r$, or

(ii) $\{y_1, \ldots, y_r\} \in \mathcal{E}(H_2)$ and $x_1 = \ldots = x_r$
Two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ with $V_1 = \{1, 2, 3, 4\}$, $\mathcal{E}_1 = (\{1, 2, 3, 4\})$
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Direct Product 1 (preserving minimal rank)

\[ \{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathcal{E}(H_1 \times_1 H_2) \]
Direct Product 1 (preserving minimal rank)

\[
\{ (x_1, y_1), \ldots, (x_r, y_r) \} \in \mathcal{E}(H_1 \times_1 H_2) \iff \begin{cases} 
\{ x_1, \ldots, x_r \} \in \mathcal{E}(H_1) \text{ and } \{ y_1, \ldots, y_r \} \subset E^2 \in \mathcal{E}(H_2), \text{ or } 
\end{cases}
\]
Direct Product 1 (preserving minimal rank)

\[ \{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathcal{E}(H_1 \times_1 H_2) \iff \]

(i) \( \{x_1, \ldots, x_r\} \in \mathcal{E}(H_1) \) and \( \{y_1, \ldots, y_r\} \subset E^2 \in \mathcal{E}(H_2) \), or

(ii) \( \{y_1, \ldots, y_r\} \in \mathcal{E}(H_2) \) and \( \{x_1, \ldots, x_r\} \subset E^1 \in \mathcal{E}(H_1) \)
Direct Product 1 (preserving minimal rank)

\[ \{(x_1, y_1), \ldots, (x_r, y_r)\} \in E(H_1 \times_1 H_2) \iff \]

(i) \( \{x_1, \ldots, x_r\} \in E(H_1) \) and \( \{y_1, \ldots, y_r\} \subset E^2 \in E(H_2) \), or

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More formal:
Direct Product 1 (preserving minimal rank)

\[ \{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathcal{E}(H_1 \times_1 H_2) \iff \]

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Edge set of the Direct Product 1 \( H = H_1 \times_1 H_2 \) of two hypergraphs \( H_1, H_2 \)
Direct Product 1 (preserving minimal rank)

\(\{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathcal{E}(H_1 \times_1 H_2) \iff\)

(i) \(\{x_1, \ldots, x_r\} \in \mathcal{E}(H_1)\) and \(\{y_1, \ldots, y_r\} \subset E^2 \in \mathcal{E}(H_2)\), or

(ii) \(\{y_1, \ldots, y_r\} \in \mathcal{E}(H_2)\) and \(\{x_1, \ldots, x_r\} \subset E^1 \in \mathcal{E}(H_1)\)

More formal:

Edge set of the Direct Product 1 \(H = H_1 \times_1 H_2\) of two hypergraphs \(H_1, H_2\):

\[
\mathcal{E}(H_1 \times_1 H_2) := \left\{ \bigcup_{x \in E} \{(x, \pi(x))\} : \pi : E \to F \text{ injective}, E \in \mathcal{E}_1, F \in \mathcal{E}_2 \right\}
\]

\[\cup \left\{ \bigcup_{y \in F} \{\pi'(y), y\} : \pi' : F \to E \text{ injective}, E \in \mathcal{E}_1, F \in \mathcal{E}_2 \right\}\]
Two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ with $V_1 = \{1, 2, 3\}, \mathcal{E}_1 = (\{1, 2, 3\})$
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Two hypergraphs $H_1 = (V_1, E_1)$ with $V_1 = \{1, 2, 3\}, E_1 = (\{1, 2, 3\})$ and $H_2 = (V_2, E_2)$ with $V_2 = \{a, b\}, E_2 = (\{a, b\})$ and their direct product $H_1 \times_1 H_2$. 
Direct Product 2 (preserving maximal rank)

\[ \{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathcal{E}(H_1 \times_2 H_2) \iff \]
Direct Product 2 (preserving maximal rank)

\[ \{(x_1, y_1), \ldots, (x_r, y_r)\} \in E(H_1 \times_2 H_2) \iff \\
(i) \{x_1, \ldots, x_r\} \in E(H_1) \text{ and } \exists E^2 \in E(H_2) \text{ s.t. } \{y_1, \ldots, y_r\} \text{ is a family of elements of } E^2, \text{ and } E^2 \subseteq \{y_1, \ldots, y_r\}, \text{ or} \]
Direct Product 2 (preserving maximal rank)

\[ \{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathcal{E}(H_1 \times_2 H_2) \iff \]

(i) \( \{x_1, \ldots, x_r\} \in \mathcal{E}(H_1) \) and \( \exists E^2 \in \mathcal{E}(H_2) \) s.t. \( \{y_1, \ldots, y_r\} \) is a family of elements of \( E^2 \), and \( E^2 \subseteq \{y_1, \ldots, y_r\} \), or

(ii) \( \{y_1, \ldots, y_r\} \in \mathcal{E}(H_2) \) and \( \exists E^1 \in \mathcal{E}(H_1) \) s.t. \( \{x_1, \ldots, x_r\} \) is a family of elements of \( E^1 \), and \( E^1 \subseteq \{x_1, \ldots, x_r\} \).
Direct Product 2 (preserving maximal rank)

\{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathcal{E}(H_1 \times_2 H_2) \iff

(i) \{x_1, \ldots, x_r\} \in \mathcal{E}(H_1) \text{ and } \exists E^2 \in \mathcal{E}(H_2) \text{ s.t. } \{y_1, \ldots y_r\} \text{ is a family of elements of } E^2, \text{ and } E^2 \subseteq \{y_1, \ldots, y_r\}, \text{ or }

(ii) \{y_1, \ldots, y_r\} \in \mathcal{E}(H_2) \text{ and } \exists E^1 \in \mathcal{E}(H_1) \text{ s.t. } \{x_1, \ldots x_r\} \text{ is a family of elements of } E^1, \text{ and } E^1 \subseteq \{x_1, \ldots, x_r\}.

More formal:
Direct Product 2 (preserving maximal rank)

\[ \{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathcal{E}(H_1 \times_2 H_2) \iff \]

(i) \( \{x_1, \ldots, x_r\} \in \mathcal{E}(H_1) \) and \( \exists E^2 \in \mathcal{E}(H_2) \) s.t. \( \{y_1, \ldots, y_r\} \) is a family of elements of \( E^2 \), and \( E^2 \subseteq \{y_1, \ldots, y_r\} \), or

(ii) \( \{y_1, \ldots, y_r\} \in \mathcal{E}(H_2) \) and \( \exists E^1 \in \mathcal{E}(H_1) \) s.t. \( \{x_1, \ldots, x_r\} \) is a family of elements of \( E^1 \), and \( E^1 \subseteq \{x_1, \ldots, x_r\} \).

More formal:
Edge set of the Direct Product 2 \( H = H_1 \times_2 H_2 \) of two hypergraphs \( H_1, H_2 \):
Direct Product 2 (preserving maximal rank)

\[ \{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathcal{E}(H_1 \times_2 H_2) \iff \]

(i) \( \{x_1, \ldots, x_r\} \in \mathcal{E}(H_1) \) and \( \exists E^2 \in \mathcal{E}(H_2) \) s.t. \( \{y_1, \ldots, y_r\} \) is a family of elements of \( E^2 \), and \( E^2 \subseteq \{y_1, \ldots, y_r\} \), or

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More formal:
Edge set of the Direct Product 2 \( H = H_1 \times_2 H_2 \) of two hypergraphs \( H_1, H_2 \):

\[
\mathcal{E}(H_1 \times H_2) := \left\{ \bigcup_{x \in E} \{(x, \varphi(x))\} \mid \varphi : E \rightarrow F \text{ surjective}, \ E \in \mathcal{E}_1, F \in \mathcal{E}_2 \right\} \\
\cup \left\{ \bigcup_{y \in F} \{(\varphi'(y), y)\} \mid \varphi' : F \rightarrow E \text{ surjective}, \ E \in \mathcal{E}_1, F \in \mathcal{E}_2 \right\}
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Two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ with $V_1 = \{1, 2, 3\}, \mathcal{E}_1 = (\{1, 2, 3\})$
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Direct Product 3

Let $E \subseteq V(H_1 \times H_2)$. $E \in \mathcal{E}(H_1 \times_3 H_2)$ if and only if

It has the form $E = \{(x, y)\} \cup ((E^1 \setminus \{x\}) \times E^2 \setminus \{y\})$

for $x \in E^1 \in \mathcal{E}(H_1)$, $y \in E^2 \in \mathcal{E}(H_2)$.
Two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ with $V_1 = \{1, 2, 3\}, \mathcal{E}_1 = (\{1, 2, 3\})$
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Two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ with $V_1 = \{1, 2, 3\}$, $\mathcal{E}_1 = (\{1, 2, 3\})$ and $H_2 = (V_2, \mathcal{E}_2)$ with $V_2 = \{a, b\}$, $\mathcal{E}_2 = (\{a, b\})$ and their Direct product $3H_1 \times 3H_2$ (In this case: $H_1 \times 3H_2 = H_1 \times 2H_2$)
More Examples

Figure: $H_1 \times_1 H_2 = H_1 \times_2 H_2$; $H_1 = (\{1,2,3\},(\{1,2,3\}))$, $H_2 = (\{a,b,c\},(\{a,b,c\}))$
More Examples

Figure: $H_1 \times_3 H_2; H_1 = ([1, 2, 3], ([1, 2, 3])), H_2 = ([a, b, c], ([a, b, c]))$
Strong Product 1

Edge set of the strong products = Edge set of the direct products $\cup$ Edge set of the Cartesian product:

1 2 3
Strong Product 1

Edge set of the strong products = Edge set of the direct products $\cup$ Edge set of the Cartesian product:

$\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}$
Strong Product 1

Edge set of the strong products = Edge set of the direct products $\cup$ Edge set of the Cartesian product:

```
1  2  3
```

Diagram:

![Diagram of strong product 1](image-url)
Strong Product 1

Edge set of the strong products = Edge set of the direct products $\cup$ Edge set of the Cartesian product:

$$\begin{align*}
1 & \quad 2 & \quad 3 \\
\end{align*}$$
Strong Product 1

Edge set of the strong products = Edge set of the direct products ∪ Edge set of the Cartesian product:

= 

\[ \begin{align*}
1a & \quad 2a & \quad 3a \\
1b & \quad 2b & \quad 3b
\end{align*} \]
Strong Products 2/3

Edge set of the strong products = Edge set of the direct products ∪ Edge set of the Cartesian product:
Strong Products 2/3

Edge set of the strong products = Edge set of the direct products $\cup$ Edge set of the Cartesian product:

1 2 3

$\Box$
Strong Products 2/3

Edge set of the strong products = Edge set of the direct products ∪ Edge set of the Cartesian product:
Strong Products 2/3

Edge set of the strong products = Edge set of the direct products $\cup$ Edge set of the Cartesian product:

\[ 1 \quad 2 \quad 3 \]
Strong Products 2/3

Edge set of the strong products $=$ Edge set of the direct products $\cup$ Edge set of the Cartesian product:

\[
\begin{align*}
1a & \quad 2a & \quad 3a \\
1b & \quad 2b & \quad 3b \\
\end{align*}
\]
Hypergraph Products

We have:

1. Restriction of the products on graphs are the common graph products
2. Associativity
3. Commutativity
4. Distributivity w.r.t. the disjoint union
5. Products of simple hypergraphs are simple
6. Unique prime factorization (under some constraints)
Hypergraph Products

We have:

1. Restriction of the products on graphs are the common graph products

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6. Unique prime factorization (under some constraints)
Hypergraph Products

We have:

1. Restriction of the products on graphs are the common graph products

2. Associativity

3. Commutativity

4. Distributivity w.r.t. the disjoint union

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Grid Property

- 2 incident edges $E, F$ of a Cartesian product belonging to two different factors span exactly one $|E| \times |F|$-grid
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\begin{tikzpicture}[scale=0.5]
    \node (E) at (0,0) {$E$};
    \node (F) at (2,2) {$F$};
    \node (u1) at (-1.5,0) {$u_1$};
    \node (v1) at (-0.5,1) {$v_1$};
    \node (z1) at (1.5,1) {$z_1$};
    \node (u2) at (-1.5,1) {$u_2$};
    \node (us) at (-1.5,3) {$u_s$};
    \path (u1) -- (v1) -- (z1) -- (u1);
    \path (u2) -- (v1) -- (z1) -- (u2);
    \path (u1) -- (u2);
    \path (v1) -- (u1);
    \path (z1) -- (u2);
\end{tikzpicture}
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$\rightarrow$ grid-property
"starting"-relation \(\delta\) on \(\mathcal{E}(H)\):

- \(E \delta F \iff\)
  
  (i) \(|E \cap F| > 1\)
  
  (ii) \(E\) and \(F\) are opposite edges of a four-cycle
  
  (iii) \(|E \cap F| = 1\) and \(\not\exists (|E| \times |F|)\)-grid without diagonals containing them.

- \(\delta^*\) suffices the grid property
"starting"-relation $\delta$ on $\mathcal{E}(H)$:

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"starting"-relation \( \delta \) on \( \mathcal{E}(H) \):

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- \( \delta^* \) suffices the grid property
• We have: relation $\delta^*$ with

$$E \delta^* F \Rightarrow E \text{ and } F \text{ belong to the same prime factor.}$$

• We want: relation $\sigma$ with

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• We have: relation $\delta^*$ with

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Equivalence classes of $\delta^*$:
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Convex hull of $\delta$, $\mathcal{C}(\delta)$

i.e. the smallest convex equivalence relation containing $\delta$:

![Graph diagram](image-url)
Convex hull of $\delta$, $\mathcal{C}(\delta)$

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Convex hull of $\delta$, $\mathcal{C}(\delta)$

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![Graph diagram]

Hypergraphs
Hypergraph Products
Prime Factorization w.r.t the Cartesian Product
Convex hull of $\delta$, $\mathcal{C}(\delta)$

i.e. the smallest convex equivalence relation containing $\delta$: 

![Diagram showing a convex hull of a set of points](image)
Convex hull of $\delta$, $C(\delta)$

i.e. the smallest convex equivalence relation containing $\delta$: 

![Diagram of a convex hull of a hypergraph]
Theorem

*Every connected Hypergraph has a unique prime factorization.*

Theorem

*The relation corresponding to the unique prime factorization of a connected hypergraph is the convex hull of the $\delta$-relation, $\sigma = C(\delta)$*
Theorem

Every connected Hypergraph has a unique prime factorization.

Theorem

The relation corresponding to the unique prime factorization of a connected hypergraph is the convex hull of the $\delta$-relation, $\sigma = \mathcal{C}(\delta)$.
Hypergraph Products

We have for the Cartesian Product

1. Restriction of the products on graphs are the common graph products √
2. Associativity √
3. Commutativity √
4. Distributivity w.r.t. the disjoint union √
5. Products of simple hypergraphs are simple √
6. Unique prime factorization (for connected hypergraphs) ?
We have for the Cartesian Product

1. Restriction of the products on graphs are the common graph products  ✓
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3. Commutativity  ✓
4. Distributivity w.r.t. the disjoint union  ✓
5. Products of simple hypergraphs are simple  ✓
6. Unique prime factorization (for connected hypergraphs)  ✓
Thanks to Marc Hellmuth and Peter Stadler!
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Thank you for your attention!