Quasi-robust Cycle Spaces.

Philipp-Jens Ostermeier
Bioinformatik Leipzig

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Motivation
A graph is an ordered 2-tuple $G = (V(G), E(G))$ with a set of vertices $V(G)$ and a set of edges $E(G) \subseteq [V(G)]^2$. 
The degree $d_v(G)$ of a vertex $v \in V(G)$ is the number of edges at $v$ in $G$. 
• A *circuit* is a connected graph $C$ with $d_v(C) = 2 \forall v \in C$. 
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• A *cycle* is an edge disjoint union of circuits.
• A *tree* is a graph in which any two vertices are connected by exactly one path.
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• A **spanning tree** $T$ of a connected graph $G$ is a tree composed of all the vertices of $G$ and $E(T) \subseteq E(G)$. 
THE EDGE SPACE

• The power set $\mathcal{E}(G)$ of $E(G)$ is an $|E(G)|$-dimensional vector space over $GF(2)$ with the:
  
  • symmetric difference $X \oplus Y := (X \cup Y) \setminus (X \cap Y)$,
  • scalar multiplication $1 \times X = X, 0 \times X = \emptyset$ for all $X, Y \in \mathcal{E}(G)$. 

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The CYCLE SPACE

The set $\mathcal{C}(G)$ of all cycles forms a subspace of $\mathcal{E}(G)$ which is called the \emph{cycle space} $\mathcal{C}(G)$. 
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Lemma

In a graph $G$ the following assertions are equivalent for edge sets $F \subseteq E(G)$:

1. $F \in \mathcal{C}(G)$;
2. $F$ is a disjoint union of (edge sets of) circuits in $G$;
3. All vertex degrees of the graph $(V(G), F)$ are even.
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2. $F$ is a disjoint union of (edge sets of) circuits in $G$;
3. All vertex degrees of the graph $(V(G), F)$ are even.

- A basis $B$ of the cycle space $\mathcal{C}(G)$ is called a *cycle basis*
The dimension $\dim_{\mathcal{C}}(G)$ of $\mathcal{C}(G)$ is the maximal number of linear independent cycles $\text{cyc}(e, T) = [e, P]$ with $P \subseteq T$, $e \in E(G) \setminus E(T)$, which is $|V(G)| - |E(G)| + 1$. 
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$G_1$  

$G_2$
Linear independent cycles.

- $\mathcal{C}(G_1 \cup G_2) = \mathcal{C}([1, 2, 3], [3, 4, 5], [6, 7, 8, 9])$
- $|\mathcal{C}(G_1 \cup G_2)| = 2^3$ and $\emptyset$ is the empty cycle.
Well-arranged sequences.

Definition
A sequence \( \mathcal{I} = (C_1, C_2, \ldots, C_k) \) of (not necessarily pairwisely distinct) cycles is well-arranged if, for each \( j \leq k \), the partial sum
\[
Q_j = \bigoplus_{i=1}^j C_i
\]
is a circuit.
The sequence \( \mathcal{I} \) is strictly well-arranged if, for \( 2 \leq j \leq k \), \( C_j \cap Q_{j-1} \) is a path.
Well-arranged sequences.

Definition
A sequence $\mathcal{S} = (C_1, C_2, \ldots, C_k)$ of (not necessarily pairwisely distinct) cycles is well-arranged if, for each $j \leq k$, the partial sum

$$Q_j = \bigoplus_{i=1}^{j} C_i$$

is a circuit.

The sequence $\mathcal{S}$ is strictly well-arranged if, for $2 \leq j \leq k$, $C_j \cap Q_{j-1}$ is a path.

In the sequence $\mathcal{S} = (C_1, \ldots, C_n)$ the cycle $C_1$ has to be a circuit.
Well-arranged Sequences of Basis Cycles.
$S = (C_1, C_2,)$ is not strictly well-arranged.
Definition

A cycle basis $\mathcal{B}$ is

- (strictly) quasi-robust if for each elementary cycle $C \in \mathcal{C}(G)$ there is a (strictly) well-arranged sequence of cycles $\mathcal{S}_C = (C_1, C_2, \ldots, C_{k_C})$ with $C_i \in \mathcal{B}, 1 \leq i \leq k_C - 1$, and $C_{k_C} = C$. 
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- The basis $\mathcal{B}$ is (strictly) robust if for each elementary cycle $C$, its (strictly) well-arranged sequence $\mathcal{I}_C$ can be chosen such that all cycles in $\mathcal{I}_C$ are linear independent.
Complete Bipartite Graphs $K_{m,n}$

$$V(K_{m,n}) = V_1 \cup V_2 \text{ with } |V_1| = m, |V_2| = n \text{ and } \{v, v'\} \in E(K_{m,n}) \text{ iff } v \in V_1, v' \in V_2$$
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The Kainen basis $B^{p,q}$

$B^{p,q} = \{ [p, q, x, y] \mid p, x \in V_1, q, y \in V_2, p, q$ are fixed $\}$ is a basis of $K_{m,n}$. 
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Definition

$\mathcal{B}$ is a strictly fundamental cycle basis of $G$

$\iff \mathcal{B} = \{ \text{cyc} (T, e) \mid e \in E(G) \setminus E(T) \}$ for some spanning tree $T$. 
A spanning tree for $B^{1,6}$.

Assume $p = 1, q = 6$. 
A spanning tree for $B^{1,6}$.

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A spanning tree for $B^{1,6}$.

Assume $p = 1$, $q = 6$.

$T^{1,6} = \left\{ \{1, 6\}, \{1, y_1\}, \ldots, \{1, y_{n-1}\}, \{6, x_1\}, \ldots, \{6, x_{m-1}\} \mid \forall x_1, \ldots, x_{n-1} \in V_1, \forall y_1, \ldots, y_{m-1} \in V_2 \right\}$
The not robust cycle basis $\mathcal{B}^{1,8}$
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Consider the circuit $\text{CYCLE} = [1, 6, 2, 7, 3, 8, 4, 9, 5, 10]$. 
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- $B^{1,8}_{\text{CYCLE}} = \{ C_{4,9}, C_{3,7}, C_{2,6}, C_{5,10}, C_{2,7}, C_{5,9} \}$, where $C_{xy} = [1, 8, x, y]$. 
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• Degree of vertex 8 is 4.
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- Degree of vertex 8 is 4.
The not robust cycle basis $\mathcal{B}^{1,8}$

$\Rightarrow$

The Kainen basis is not robust.
Theorem
The Kainen basis $\mathcal{K}$ of $K_{m,n}$ is quasi-robust for all $m, n$. 
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Let us construct a quasi-robust sequence $\mathcal{I}$ for $\text{CYCLE} = [1, 6, 2, 7, 3, 8, 4, 9, 5, 10]$. 
Consider the circuit $\text{CYCLE} = [1, 6, 2, 7, 3, 8, 4, 9, 5, 10]$.

Let $\mathcal{L} = (C_{4,6}, C_{4,9}, C_{5,9}, C_{2,6}, C_{2,7}, C_{5,10}, C_{3,7}, C_{4,6})$, where $C_{xy} = [1, 8, x, y]$. 
Consider the circuit $\text{CYCLE} = [1, 6, 2, 7, 3, 8, 4, 9, 5, 10]$.

- $\mathcal{S} = (C_{4,6}, C_{4,9}, C_{5,9}, C_{2,6}, C_{2,7}, C_{5,10}, C_{3,7}, C_{4,6})$, where $C_{xy} = [1, 8, x, y]$. 

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Consider the circuit \( \text{CYCLE} = [1, 6, 2, 7, 3, 8, 4, 9, 5, 10] \).

\( \mathcal{I} = (C_{4,6}, C_{4,9}, C_{5,9}, C_{2,6}, C_{2,7}, C_{5,10}, C_{3,7}, C_{4,6}), \) where \( C_{xy} = [1, 8, x, y] \).
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Is every cycle basis quasi-robust?

- $T = \{\text{black edges}\}$
  $\Rightarrow B_T = \{C_1, \ldots, C_6\}$
- $C = \bigoplus_{i=1}^{6} C_i$ is the green circuit.
- $C \oplus C_i, \ i = 1, \ldots, 6$ is no circuit

**ANSWER:** No, and not every strictly fundamental cycle basis is quasi-robust.
Theorem
Every triangular strictly fundamental cycle basis is strictly robust.
**Theorem**

Every triangular strictly fundamental cycle basis is strictly robust.

Example: $K_5$, the graph with 5 vertices in which all vertices are pairwise connected.
The red spanning tree $T$ induces a triangular basis with 6 elements.
Consider the green cycle, $C \cap T = \emptyset$. 

$K_5$
Basics. The Cycle Space

(Quasi-)Robust Cycle Bases

\[ \mathbf{C} \oplus [1, 2, 3]. \]
• $C \oplus [1, 2, 3] \oplus [1, 3, 4]$. 

$K_5$
Basics. The Cycle Space

(Quasi-)Robust Cycle Bases

• $C \oplus [1, 2, 3] \oplus [1, 3, 4] \oplus [1, 5, 4]$. 

$K_5$
$C \oplus [1, 2, 3] \oplus [1, 3, 4] \oplus [1, 5, 4] \oplus [1, 2, 5]$. 
• Consider the green cycle, $C' \cap T \neq \emptyset$. 
Basics. The Cycle Space

(Quasi-)Robust Cycle Bases

$K_5$

- $C' \oplus [1, 2, 5] = C$, now we can start as above.
Is every triangular cycle basis robust?

- Consider the cycle basis $B$ which consists of all outer triangles and the triangle $[1,3,4]$. The symmetric difference of all the outer triangles is the red coloured star $S$. 

![Diagram](image-url)
Is every triangular cycle basis robust?

$S \oplus [0, 1, 2]$ is no circuit which holds for all triangles. So $\mathcal{B}$ is not robust.
The Cartesian product.

\[ K_2 \square C \]

\{ (0, 0'), (1, 0') \} is an edge in \( K_2 \square C \) because \( \{0, 1\} \in E(K_2) \) and \( 0' = 0' \).
Is every cycle basis quasi-robust?

- The blue graph $G$ has the vertex set $V(G) = \{1, \ldots, 9\}$.
- Its black copy $G'$ has the vertex set $V(G') = \{1', \ldots, 9'\}$.
Is every cycle basis quasi-robust?

Consider the circuit
\[ C = [4, 2, 7, 7', 3', 9', 9, 1, 6, 6', 2', 5', 5, 3, 8, 8', 1', 4'].\]
Is every cycle basis quasi-robust?

C ⊕ [5, 7, 3] has degree 4 at 7.
Is every cycle basis quasi-robust?

\[ C \oplus [2, 1, 3] \text{ has degree 4 at } 2. \]
Is every cycle basis quasi-robust?

$C \oplus [2, 2', 5', 5]$ has degree 4 at 2.
Thanks for your attention!

Thanks to Peter, Konstantin, Marc, Dieter, Nancy...