Span of a graph: keeping the safety distance

Iztok Banič*1,2,3 and Andrej Taranenko†1,2

1University of Maribor, Faculty of Natural Sciences and Mathematics, Koroška cesta 160, SI-2000 Maribor, Slovenia
2Institute of Mathematics, Physics and Mechanics, Jadranska 19, SI-1000 Ljubljana, Slovenia
3Andrej Marušič Institute, University of Primorska, Muzejski trg 2, SI-6000 Koper, Slovenia

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Abstract

Inspired by Lelek’s idea from [2], we introduce the notion of span to the theory of graphs. Using this, we solve the keeping the maximum safety distance problem. This problem is a graph traversal problem in which two visitors must visit all vertices/edges of a graph whilst maintaining the maximal possible distance. Moreover, their moves must be made with respect to certain move rules.

Keywords: strong span of a graph, direct span of a graph, Cartesian span of a graph, safety distance

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1 Introduction

In the times of a global pandemic which we have witnessed starting in 2020, two of the basic public safety measures that were introduced worldwide were social distancing and keeping a safety distance in public spaces. In this paper, we present a mathematical model for computing a minimum safety distance two travellers can keep at all times. Our model is based on Lelek’s span of a continuum which is introduced in [2].

Imagine two visitors, say $A$ and $B$, in a gallery at the same time. They would both like to see all exhibits whilst keeping the maximum possible safety distance from each other. The exhibits can be assumed to be inside rooms (represented by vertices of a graph). One way to describe the visitor’s tours is to represent...
their positions at a fixed moment $t$ in time by a pair $(a_t, b_t)$, where both $a_t$ and $b_t$ are vertices of the graph. After that either person can choose to move to the next room or stay at the current position. Of course, each may only move to an adjacent room.

Figure 1 shows an example of a floor plan of a gallery represented by the graph $G$, and the location (the walk) of both visitors at five consecutive points in time, shown by the graph $W$. For each point in time the location of visitor $A$ is represented by a mapping with a red arrow and the blue arrows map to the location of visitor $B$. At the time $t_0$, visitor $A$ is in the room $r_1$ and visitor $B$ is in the room $r_3$, so their locations can be represented by the pair $(r_1, r_3)$. In this case the visitors are keeping safe at the distance 2. Next, at the time $t_1$, visitor $A$ moves to the room $r_2$, whilst the visitor $B$ remains put in the room $r_3$ (this can be represented by the pair $(r_2, r_3)$, and the visitors are at the distance 1). At the time $t_2$, the visitor $A$ moves to the room $r_3$ and the visitor $B$ moves to the room $r_4$; thus obtaining the pair $(r_3, r_4)$ and maintaining the safety distance 1. Similarly, at the time $t_3$, the visitor $A$ stays in the room $r_3$ and the visitor $B$ moves to the room $r_2$; thus obtaining the pair $(r_3, r_2)$ and maintaining the safety distance 1. Finally, at the time $t_4$, visitor $A$ moves to the room $r_4$, whilst the visitor $B$ moves to the room $r_1$ (this can be represented by the pair $(r_4, r_1)$, and the visitors are at the distance 2). So in this example the walk of both visitors at five consecutive points in times can be represented by the tuple $((r_1, r_3), (r_2, r_3), (r_3, r_4), (r_3, r_2), (r_4, r_1))$. Note, in this example all rooms are visited by both visitors and the maximum distance they were able to maintain at all times was one.

![Figure 1](image.png)

Figure 1: An example of two visitors’ walks in a gallery.

We are interested in keeping the maximum possible safety distance between both visitors amongst all possible walks through the gallery. Moreover, in our
model we assume that both desire to visit all rooms (all vertices should be visited) and/or also, in cases where the hallways also include exhibits, all hallways (all edges should be visited). In what follows, we present a formal model for the described situations for three different sets of movement rules at any observed point in time:

- both visitor are independently allowed to move to an adjacent room or stay at their current locations,
- exactly one visitor is allowed to move to an adjacent room, or
- both visitors must move to an adjacent room.

We proceed as follows. In Section 2 basic definitions and notations are presented. In Section 3 we define different vertex and edge span variants of a graph and prove that each span can be obtained from a corresponding graph product. We continue with Section 4, where we characterise 0-span graphs for each variant. Moreover, we present an infinite family of graphs for which the vertex and edge variant of the corresponding span are equal. We conclude the paper with several open problems.

2 Preliminary results

Our terminology and notation mostly follow [1].

**Definition 2.1.** A graph $G$ is an ordered pair $(V(G), E(G))$, where $V(G)$ is any non-empty finite set, called the set of vertices of the graph $G$, and $E(G)$ is any subset of the set $\{\{u,v\} \mid u,v \in V(G) \text{ and } u \neq v\}$ and it is called the set of edges of the graph $G$.

For any graph $G$ and any edge $\{u,v\} \in E(G)$ we sometimes use a much simpler notation $uv$ to denote the edge $\{u,v\}$.

**Definition 2.2.** Let $n$ be a positive integer, $G = (V(G), E(G))$ be a graph and let $u, v \in V(G)$. The $n$-tuple $(v_1, v_2, v_3, \ldots, v_n) \in V(G)^n$ is a path in the graph $G$ from $u$ to $v$, if

1. $v_1 = u$, $v_n = v$,
2. $|\{v_1, v_2, v_3, \ldots, v_n\}| = n$, and
3. $v_i v_{i+1} \in E(G)$ for any positive integer $i \in \{1, 2, \ldots, n-1\}$.

The length of such a path $(v_1, v_2, v_3, \ldots, v_n)$ is $n - 1$.

**Definition 2.3.** The distance $d_G(u, v)$ in a graph $G$ from a vertex $u \in V(G)$ to a vertex $v \in V(G)$ is the length of a shortest path in $G$ from $u$ to $v$. If there is no path from $u$ to $v$, then we define $d_G(u, v) = \infty$ (we also define that for each non-negative integer $k$, $k$ is less than $\infty$).
Definition 2.4. A graph $G$ is connected if for any pair $(u, v) \in V(G)^2$ there is a path in $G$ from $u$ to $v$.

Definition 2.5. Let $G$ be a connected graph and $v$ be a vertex of $G$. The eccentricity of the vertex $v$, denoted $\text{ecc}(v)$, is the maximum distance from $v$ to any vertex of $G$. That is,

$$\text{ecc}(v) = \max\{d_G(v, u) \mid u \in V(G)\}.$$  

The radius of $G$, denoted $\text{rad}(G)$, is the minimum eccentricity among the vertices of $G$. Therefore,

$$\text{rad}(G) = \min\{\text{ecc}(v) \mid v \in V(G)\}.$$  

The diameter of $G$, denoted $\text{diam}(G)$, is the maximum eccentricity among the vertices of $G$, thus,

$$\text{diam}(G) = \max\{\text{ecc}(v) \mid v \in V(G)\}.$$  

Definition 2.6. For any positive integer $n$ we use $K_n$ to denote the complete graph on $n$ vertices, i.e. the graph on $n$ vertices in which any two distinct vertices are adjacent.

Definition 2.7. For graphs $G$ and $H$ we will use the notation $G \subseteq H$ to denote that $G$ is a subgraph of $H$, meaning that $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. Moreover, $G \subseteq_C H$ denotes that $G$ is a connected subgraph of $H$.

Definition 2.8. Let $H$ be a graph and let $G \subseteq H$. The graph $H - G$ is defined by $V(H - G) = V(H) \setminus V(G)$ and $E(H - G) = E(H) \setminus \{uv \in E(H) \mid u \in V(G)\}$.

Definition 2.9. Let $H$ be a graph. A graph $K \subseteq H$ is a component of $H$, if $K$ is connected and for any connected graph $G \subseteq H$ it holds that

$$V(G) \cap V(K) \neq \emptyset \implies G \subseteq K.$$  

For a set $U$ of vertices of a graph $G$ we denote by $\langle U \rangle_G$ the subgraph of $G$ induced by the set $U$; see [1] for the definition. The index $G$ may be omitted when the graph will be clear from the context.

Definition 2.10. Let $G$ and $H$ be any graphs. A function $f : V(G) \to V(H)$ is a weak homomorphism from $G$ to $H$ if for all $u, v \in V(G)$, $uv \in E(G)$ implies $f(u)f(v) \in E(H)$ or $f(u) = f(v)$. We will use the more common notation $f : G \to H$ to say that $f : V(G) \to V(H)$ is a weak homomorphism.

Definition 2.11. A weak homomorphism $f : G \to H$ is surjective if $f(V(G)) = V(H)$. A weak homomorphism $f : G \to H$ is edge surjective if it is surjective and for every $uv \in E(H)$ there exists an edge $xy \in E(G)$ such that $u = f(x)$ and $v = f(y)$. 

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Definition 2.12. Let $f : G \to H$ be a weak homomorphism from $G$ to $H$ and let $K \subseteq G$. The restriction $f|_K : K \to H$ is defined by $f|_K(u) = f(u)$ for any $u \in V(K)$.

Definition 2.13. Let $f : G \to H$ be a weak homomorphism. The image $f(G)$ of the graph $G$ is defined by:

$V(f(G)) = \{f(u) \mid u \in V(G)\}$,

$E(f(G)) = \{f(u)f(v) \mid uv \in E(G) \text{ and } f(u) \neq f(v)\}$.

Note that if $f : G \to H$ is a weak homomorphism, then $f(G) \subseteq H$. The following lemma is a well-known result. Since the proof is short, we give it anyway.

Lemma 2.14. Let $f : G \to H$ be a weak homomorphism. If $G$ is a connected graph, then also $f(G)$ is a connected graph.

Proof. Note that since $f : G \to H$ is a weak homomorphism, it follows that

$d_{f(G)}(f(u), f(v)) \leq d_G(u, v)$

for any $u, v \in V(G)$. Therefore, there is a path in $f(G)$ from $f(u)$ to $f(v)$ for any $u, v \in V(G)$. □

Definition 2.15. Let $G$ and $H$ be any graphs. In the present paper, we deal with the following three products of $G$ and $H$:

1. The Cartesian product $G \square H$ is defined by

$V(G \square H) = V(G) \times V(H)$

and

$E(G \square H) = \{(u_1, v_1)(u_2, v_2) \mid u_1 = u_2 \text{ and } v_1v_2 \in E(H) \text{ or } u_1u_2 \in E(G) \text{ and } v_1 = v_2\}.$

2. The strong product $G \boxtimes H$ is defined by

$V(G \boxtimes H) = V(G) \times V(H)$

and

$E(G \boxtimes H) = \{(u_1, v_1)(u_2, v_2) \mid [u_1u_2 \in E(G) \text{ and } v_1 = v_2 \text{ or } v_1v_2 \in E(H)] \text{ or } [v_1v_2 \in E(H) \text{ and } u_1 = u_2 \text{ or } u_1u_2 \in E(G)]\}$.
3. The direct product $G \times H$ is defined by

$$V(G \times H) = V(G) \times V(H)$$

and

$$E(G \times H) = \{(u_1, v_1)(u_2, v_2) \mid u_1u_2 \in E(G) \text{ and } v_1v_2 \in E(H)\}.$$ 

**Definition 2.16.** Let $G$ and $H$ be any graphs. The functions

$$p_1 : V(G) \times V(H) \to V(G) \text{ and } p_2 : V(G) \times V(H) \to V(H),$$

defined by $p_1(u, v) = u$ and $p_2(u, v) = v$ for each $(u, v) \in V(G) \times V(H)$ are called the first and the second projection functions, respectively. We also refer to them as the projection functions or simply, the projections.

**Observation 2.17.** Let $G$ and $H$ be any graphs. It follows from

$$V(G) \times V(H) = V(G \times H) = V(G \sqcup H) = V(G \Join H)$$

that both projections

$$p_1 : V(G) \times V(H) \to V(G) \text{ and } p_2 : V(G) \times V(H) \to V(H)$$

are also weak homomorphisms

$$p_1 : G \times H \to G \text{ and } p_2 : G \times H \to H,$$

$$p_1 : G \sqcup H \to G \text{ and } p_2 : G \sqcup H \to H$$

or

$$p_1 : G \Join H \to G \text{ and } p_2 : G \Join H \to H.$$

At the end of the section, we define a distance between two homomorphisms, which will be used in Section 3 to introduce all the variants of the spans of graphs.

**Definition 2.18.** Let $f, g : G \to H$ be weak homomorphisms. We define

$$m_G(f, g) = \min\{d_H(f(u), g(u)) \mid u \in V(G)\}$$

to be the distance from $f$ to $g$.

**Observation 2.19.** Let $f, g : G \to H$ be weak homomorphisms. Note that

$$m_G(f, g) \leq \text{diam}(H),$$

if $G$ is connected. If $G$ is not connected, then $m_G(f, g)$ may equal $\infty$. 

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Lemma 2.20. If $f, g : G \to H$ are surjective weak homomorphisms and $G$ is connected, then

$$m_G(f, g) \leq \text{rad}(H).$$

Proof. Let $G$ be a connected graph and $f, g : G \to H$ be surjective weak homomorphisms. Let $u \in V(H)$ be such that $\text{ecc}(u) = \text{rad}(H)$, i.e. $u$ is a vertex of $H$ with eccentricity equal to the radius of $H$. Since $f$ is surjective, there is a vertex $v \in V(G)$ such that $f(v) = u$. Therefore $d_H(f(v), g(v)) \leq \text{rad}(H)$. This implies that $m_G(f, g) \leq \text{rad}(H)$. □

Definition 2.21. Let $H$ be a connected graph and let $Z$ be a graph such that $V(Z) \subseteq V(H) \times V(H)$. We define

$$\varepsilon_H(Z) = \min\{d_H(x, y) \mid (x, y) \in V(Z)\}.$$

3 Span - definitions and basic properties

Here we introduce six different variants of a span of a given connected graph: the strong edge span, the strong vertex span, the direct edge span, the direct vertex span, the Cartesian edge span, and the Cartesian vertex span.

All the variants model the maximum safety distance kept by two travellers in graph traversal, where the travellers may only move with accordance to a specific set of rules. Moreover, for each set of rules we define the vertex and the edge span variant. In the first, all vertices of the graph must be visited at least once by both travellers (the vertex variant of a span), and in the second all vertices and edges must be traversed by both at least once (the edge variant of a span).

3.1 Strong span

For the strong span variant the rules of movement are the following: at any moment in time any of the travellers may either stay at the same vertex, or move to an adjacent one. These rules can be described using weak homomorphisms.

We now define the strong edge and the strong vertex span of a graph.

Definition 3.1. Let $H$ be a connected graph. Define

$$\sigma_E^S(H) = \max\{m_G(f, g) \mid f, g : G \to H \text{ are edge surjective}$$

$$\text{weak homomorphisms and } G \text{ is connected}\}.$$ 

We call $\sigma_E^S(H)$ the strong edge span of the graph $H$.

Define

$$\sigma_V^S(H) = \max\{m_G(f, g) \mid f, g : G \to H \text{ are surjective}$$

$$\text{weak homomorphisms and } G \text{ is connected}\}.$$ 

We call $\sigma_V^S(H)$ the strong vertex span of the graph $H$. 

Observation 3.2. Note that for any connected graph $H$, the sets

\[ \{m_G(f,g) \mid f, g: G \to H \text{ are edge surjective weak homomorphisms and } G \text{ is connected} \} \]

and

\[ \{m_G(f,g) \mid f, g: G \to H \text{ are surjective weak homomorphisms and } G \text{ is connected} \} \]

are non-empty subsets of non-negative integers and are bounded from above by $\text{rad}(H)$. Therefore, $\sigma^{\mathbb{E}}_{V}(H)$ and $\sigma^{\mathbb{E}}_{E}(H)$ are well-defined. Note also that

\[ \sigma^{\mathbb{E}}_{E}(H) \leq \sigma^{\mathbb{E}}_{V}(H) \leq \text{rad}(H). \]

For any connected graph $H$, the following two theorems show that it is not necessary to consider all corresponding weak homomorphisms from all possible connected graphs $G$. Instead it suffices to consider only connected subgraphs of $H \boxtimes H$ and projections $p_1, p_2 : H \boxtimes H \to H$.

Theorem 3.3. If $H$ is a connected graph, then

\[ \sigma^{\mathbb{E}}_{V}(H) = \max \{\varepsilon_{H}(Z) \mid Z \subseteq C \text{ } H \boxtimes H \text{ with } p_1(V(Z)) = p_2(V(Z)) = V(H) \}. \]

Proof. We define

\[ A = \{\varepsilon_{H}(Z) \mid Z \subseteq C \text{ } H \boxtimes H \text{ with } p_1(V(Z)) = p_2(V(Z)) = V(H) \} \]

and

\[ B = \{m_G(f,g) \mid f, g: G \to H \text{ are surjective weak homomorphisms and } G \text{ is connected} \}. \]

We will show that $\max(A) = \max(B)$ by proving that $A = B$. First we show that $A \subseteq B$. Let $r \in A$ be arbitrary. Let $Z$ be a connected subgraph of $H \boxtimes H$ such that $p_1(V(Z)) = p_2(V(Z)) = V(H)$ and $\varepsilon_{H}(Z) = r$. Let $G = Z$, $f = p_1|_G$ and $g = p_2|_G$. Then

1. $G$ is a connected graph,
2. $f, g : G \to H$ are surjective weak homomorphisms, and
3. 

\[ m_G(f,g) = \min \{d_H(f(u),g(u)) \mid u \in V(G) \} \]
\[ = \min \{d_H(p_1(u),p_2(u)) \mid u \in V(Z) \} \]
\[ = \min \{d_H(x,y) \mid (x,y) \in V(Z) \} \]
\[ = \varepsilon_{H}(Z) = r. \]
Therefore, \( r \in B \) and we have proved that \( A \subseteq B \).

To show that \( B \subseteq A \), let \( r \in B \) be arbitrary. Let \( G \) be a connected graph and let \( f, g : G \to H \) be surjective weak homomorphisms such that \( m_G(f, g) = r \). Define \( \psi : V(G) \to V(H \boxtimes H) \) by \( \psi(u) = (f(u), g(u)) \) for all \( u \in V(G) \). We claim that \( \psi \) is a well-defined weak homomorphism. It is obvious that \( \psi(u) \in V(H \boxtimes H) \) for any \( u \in V(G) \). Let \( uv \in E(G) \). The following cases are possible:

1. \( f(u) = f(v) \) and \( g(u) = g(v) \). Here \( \psi(u) = \psi(v) \).

2. \( f(u)f(v) \in E(H) \) and \( g(u) = g(v) \). Here \( \psi(u) = (f(u), g(u)) = (f(u), g(v)) \) and \( \psi(v) = (f(v), g(v)) \). Therefore \( \psi(u)\psi(v) \in E(H \boxtimes H) \).

3. \( f(u) = f(v) \) and \( g(u)g(v) \in E(H) \). Here \( \psi(u) = (f(u), g(u)) = (f(v), g(u)) \) and \( \psi(v) = (f(v), g(v)) \). Therefore \( \psi(u)\psi(v) \in E(H \boxtimes H) \).

4. \( f(u)f(v) \in E(H) \) and \( g(u)g(v) \in E(H) \). Here \( \psi(u) = (f(u), g(u)) \) and \( \psi(v) = (f(v), g(v)) \). Therefore \( \psi(u)\psi(v) \in E(H \boxtimes H) \).

It follows that \( \psi \) is a well-defined weak homomorphism from \( G \) to \( H \boxtimes H \). Let \( Z = \psi(G) \). Since \( G \) is connected, it follows that \( Z \) is a connected subgraph of \( H \boxtimes H \). Next we show that \( p_1(V(Z)) = V(H) \) and \( p_2(V(Z)) = V(H) \). Let \( x \in V(H) \). Since \( f \) and \( g \) are surjective weak homomorphisms, there are \( u, v \in V(G) \) such that \( f(u) = x \) and \( g(v) = x \). Then \( p_1(f(u), g(u)) = x \) and \( p_2(f(v), g(v)) = x \) and we are done.

Since

\[
\varepsilon_H(Z) = \min \{ d_H(x, y) \mid (x, y) \in V(Z) \} = \min \{ d_H(f(u), g(u)) \mid u \in V(G) \} = m_G(f, g) = r,
\]

it follows that \( r \in A \). Hence, \( B \subseteq A \) and we have proved that \( A = B \). It follows that \( \max(A) = \max(B) \).

\( \square \)

**Theorem 3.4.** Let \( H \) be a connected graph. Then

\[
\sigma^H(H) = \max \{ \varepsilon_H(Z) \mid Z \subseteq C \ H \boxtimes H \text{ with } p_1(Z) = p_2(Z) = H \}.
\]

**Proof.** We define

\[
A = \{ \varepsilon_H(Z) \mid Z \subseteq C \ H \boxtimes H \text{ with } p_1(Z) = p_2(Z) = H \} \text{ and }
\]

\[
B = \{ m_G(f, g) \mid f, g : G \to H \text{ are edge surjective weak homomorphisms and } G \text{ is connected} \}.
\]

Similarly to the proof of Theorem 3.3 we prove the assertion by proving that \( A = B \). To show that \( A \subseteq B \), let \( r \in A \) be arbitrary. Let \( Z \subseteq C \ H \boxtimes H \) be such that \( p_1(Z) = p_2(Z) = H \) and \( \varepsilon_H(Z) = r \). Let \( G = Z, f = p_1|_Z \) and \( g = p_2|_Z \). Then

1. \( G \) is a connected graph,
2. $f, g : G \to H$ are edge surjective weak homomorphisms, and

3. 

$$m_G(f, g) = \min\{d_H(f(u), g(u)) \mid u \in V(G)\} = \min\{d_H(p_1(u), p_2(u)) \mid u \in V(Z)\} = \min\{d_H(x, y) \mid (x, y) \in V(Z)\} = \varepsilon_H(Z) = r.$$ 

Therefore, $r \in B$ and we have proved that $A \subseteq B$. To show that $B \subseteq A$, let $r \in B$. Let $G$ be a connected graph and let $f, g : G \to H$ be edge surjective weak homomorphisms such that $m_G(f, g) = r$. Let $Z$ be the graph defined by

$$V(Z) = \{(f(u), g(u)) \mid u \in V(G)\}$$

and, for any two vertices $(u, v)$ and $(u', v')$ of the graph $Z$, $(u, v)(u', v') \in E(Z)$ if and only if one of the following three conditions is satisfied:

1. $uu' \in E(H)$ and $v = v'$, or
2. $vv' \in E(H)$ and $u = u'$, or
3. $uu' \in E(H)$ and $vv' \in E(H)$.

Define $\psi : V(G) \to V(Z)$ by $\psi(u) = (f(u), g(u))$ for all $u \in V(G)$. Note that $\psi$ is an edge surjective weak homomorphism from $G$ to $Z$. Therefore, $Z = \psi(G)$ and since $G$ is connected, by Lemma 2.14 the graph $Z$ is also connected. Therefore, $Z \subseteq C H \bowtie H$. Since

$$\varepsilon_H(Z) = \min\{d_H(x, y) \mid (x, y) \in V(Z)\} = \min\{d_H(f(u), g(u)) \mid u \in V(G)\} = m_G(f, g) = r,$$

it follows that $r \in A$. Hence, $B \subseteq A$ and we have proved that $A = B$. It follows that $\max(A) = \max(B)$. $\square$

### 3.2 Direct span

For the direct span variant the rules of movement are the following: at any moment in time both travellers must move to an adjacent vertex. This rule can be described using aligned weak homomorphisms.

We now define the direct edge and the direct vertex span of a graph.

**Definition 3.5.** Let $f, g : G \to H$ be weak homomorphisms. We say that $f$ and $g$ are aligned weak homomorphisms, if for any $uv \in E(G)$,

$$f(u)f(v) \in E(H) \iff g(u)g(v) \in E(H).$$
Definition 3.6. Let \( H \) be a connected graph. Define
\[
\sigma_E^\times(H) = \max\{m_G(f, g) \mid f, g : G \to H \text{ are edge surjective aligned weak homomorphisms and } G \text{ is connected}\}.
\]
We call \( \sigma_E^\times(H) \) the direct edge span of the graph \( H \).
Define
\[
\sigma_V^\times(H) = \max\{m_G(f, g) \mid f, g : G \to H \text{ are surjective aligned weak homomorphisms and } G \text{ is connected}\}.
\]
We call \( \sigma_V^\times(H) \) the direct vertex span of the graph \( H \).

Observation 3.7. Note that for any connected graph \( H \),
\[
\sigma_E^\times(H) \leq \sigma_V^\times(H) \leq \text{rad}(H).
\]
For any connected graph \( H \), the following theorem shows that it is not necessary to consider all corresponding weak homomorphisms from all possible connected graphs \( G \). Instead it suffices to consider only connected subgraphs of \( H \times H \) and projections \( p_1, p_2 : H \times H \to H \).

Theorem 3.8. If \( H \) is a connected graph, then
\[
\sigma_V^\times(H) = \max\{\varepsilon_H(Z) \mid Z \subseteq C \ H \times H \text{ with } p_1(V(Z)) = p_2(V(Z)) = V(H)\}
\]
and
\[
\sigma_E^\times(H) = \max\{\varepsilon_H(Z) \mid Z \subseteq C \ H \times H \text{ with } p_1(Z) = p_2(Z) = H\}.
\]

Proof. Denote by
\[
A = \{\varepsilon_H(Z) \mid Z \subseteq C \ H \times H \text{ with } p_1(V(Z)) = p_2(V(Z)) = V(H)\},
\]
\[
B = \{\varepsilon_H(Z) \mid Z \subseteq C \ H \times H \text{ with } p_1(Z) = p_2(Z) = H\},
\]
\[
C = \{m_G(f, g) \mid f, g : G \to H \text{ are surjective aligned weak homomorphisms and } G \text{ is connected}\} \text{ and}
\]
\[
D = \{m_G(f, g) \mid f, g : G \to H \text{ are edge surjective aligned weak homomorphisms and } G \text{ is connected}\}.
\]
We prove that \( \max(A) = \max(C) \) and \( \max(B) = \max(D) \) by showing that \( A = C \) and \( B = D \).

The proof that \( A \subseteq C \) is analogous to the proof of Theorem 3.3 (the part that \( A \subseteq B \)). Also, the proof that \( B \subseteq D \) is analogous to the proof of Theorem 3.3 (the part that \( A \subseteq B \)).

To show that \( C \subseteq A \), let \( r \in C \) be arbitrary. Let \( G \) be a connected graph and let \( f, g : G \to H \) be surjective aligned weak homomorphisms such that \( m_G(f, g) = r \). Let \( Z \) be the graph defined by
\[
V(Z) = \{(f(u), g(u)) \mid u \in V(G)\}
\]
and, for any two vertices \((u, v)\) and \((u', v')\) of the graph \(Z, (u, v)(u', v') \in E(Z)\) if and only if \(uu' \in E(H)\) and \(vv' \in E(H)\). Note that \(Z \subseteq H \times H\).

Define \(\psi : V(G) \to V(Z)\) by \(\psi(u) = (f(u), g(u))\) for all \(u \in V(G)\). Note that \(\psi\) is a surjective weak homomorphism from \(G\) to \(Z\). Since \(G\) is connected, also \(Z\) is connected by Lemma \(2.14\). Therefore, \(Z \subseteq C \times H \times H\).

Since \(Z\) is connected, also \(Z\) is connected. Therefore, \(Z \subseteq C \times H \times H\). Since

\[
\varepsilon_H(Z) = \min\{d_H(x, y) \mid (x, y) \in V(Z)\} = \min\{d_H(f(u), g(u)) \mid u \in V(G)\}
\]

\[
= m_G(f, g) = r,
\]

it follows that \(r \in A\). Hence, \(C \subseteq A\) and we have proved that \(A = C\).

Finally, to show that \(D \subseteq B\), let \(r \in D\) be arbitrary. Let \(G\) be a connected graph and let \(f, g : G \to H\) be edge surjective aligned weak homomorphisms such that \(m_G(f, g) = r\). Let \(Z\) be the graph defined by

\[
V(Z) = \{(f(u), g(u)) \mid u \in V(G)\}
\]

and, for any two vertices \((u, v)\) and \((u', v')\) of the graph \(Z, (u, v)(u', v') \in E(Z)\) if and only if \(uu' \in E(H)\) and \(vv' \in E(H)\). Define \(\psi : V(G) \to V(Z)\) by \(\psi(u) = (f(u), g(u))\) for all \(u \in V(G)\). Note that \(\psi\) is an edge surjective weak homomorphism from \(G\) to \(Z\). Therefore, \(Z = \psi(G)\) and since \(G\) is connected, by Lemma \(2.14\) the graph \(Z\) is also connected. Therefore, \(Z \subseteq C \times H \times H\). Since

\[
\varepsilon_H(Z) = \min\{d_H(x, y) \mid (x, y) \in V(Z)\} = \min\{d_H(f(u), g(u)) \mid u \in V(G)\}
\]

\[
= m_G(f, g) = r,
\]

it follows that \(r \in B\). Hence, \(D \subseteq B\) and we have proved that \(B = D\). 

\[\square\]

### 3.3 Cartesian span

For the Cartesian span variant the rules of movement are the following: at any moment in time exactly one of the travellers must move to an adjacent vertex. These rules can be described using opposite weak homomorphisms.

We now define the Cartesian edge and the Cartesian vertex span of a graph.

**Definition 3.9.** Let \(f, g : G \to H\) be weak homomorphisms. We say that \(f\) and \(g\) are opposite weak homomorphisms, if for any \(uv \in E(G)\),

\[
f(u)f(v) \in E(H) \iff g(u) = g(v).
\]

**Definition 3.10.** Let \(H\) be a connected graph. Define

\[
\sigma^E_H = \max\{m_G(f, g) \mid f, g : G \to H\text{ are edge surjective opposite weak homomorphisms and }G\text{ is connected}\}.
\]

We call \(\sigma^E_H\) the Cartesian edge span of the graph \(H\).
Define
\[ \sigma_V^\Box(H) = \max \{ m_G(f, g) \mid f, g : G \to H \text{ are surjective opposite weak homomorphisms and } G \text{ is connected} \} . \]

We call \( \sigma_V^\Box(H) \) the Cartesian vertex span of the graph \( H \).

**Observation 3.11.** Note that for any connected graph \( H \),
\[ \sigma_E^\Box(H) \leq \sigma_V^\Box(H) \leq \text{rad}(H). \]

For any connected graph \( H \), the following theorem shows that it is not necessary to consider all corresponding weak homomorphisms from all possible connected graphs \( G \). Instead it suffices to consider only connected subgraphs of \( H \Box H \) and projections \( p_1, p_2 : H \Box H \to H \).

**Theorem 3.12.** If \( H \) is a connected graph, then
\[ \sigma_V^\Box(H) = \max \{ \varepsilon_H(Z) \mid Z \subseteq C \text{ } H \Box H \text{ with } p_1(V(Z)) = p_2(V(Z)) = V(H) \} \]
and
\[ \sigma_E^\Box(G) = \max \{ \varepsilon_H(Z) \mid Z \subseteq C \text{ } H \Box H \text{ with } p_1(Z) = p_2(Z) = H \} , \]

**Proof.** Denote by
\begin{align*}
A &= \{ \varepsilon_H(Z) \mid Z \subseteq C \text{ } H \Box H \text{ with } p_1(V(Z)) = p_2(V(Z)) = V(H) \} , \\
B &= \{ \varepsilon_H(Z) \mid Z \subseteq C \text{ } H \Box H \text{ with } p_1(Z) = p_2(Z) = H \} , \\
C &= \{ m_G(f, g) \mid f, g : G \to H \text{ are surjective opposite weak homomorphisms and } G \text{ is connected} \} \text{ and} \\
D &= \{ m_G(f, g) \mid f, g : G \to H \text{ are edge surjective opposite weak homomorphisms and } G \text{ is connected} \} .
\end{align*}

As in the proofs of Theorems 3.3, 3.4, and 3.8, we prove that \( \max(A) = \max(C) \) and \( \max(B) = \max(D) \) by showing that \( A = C \) and \( B = D \).

The first part of the proof (that \( A \subseteq C \) and \( B \subseteq D \) follows the same line of thought as in the proofs Theorems 3.3, 3.4, and 3.8.

To show that \( C \subseteq A \), let \( r \in C \). Let \( G \) be a connected graph and let \( f, g : G \to H \) be surjective opposite weak homomorphisms such that \( m_G(f, g) = r \). Let \( Z \) be the graph defined by
\[ V(Z) = \{(f(u), g(u)) \mid u \in V(G) \} \]
and, for any two vertices \((u, v)\) and \((u', v')\) of the graph \( Z \), \((u, v)(u', v') \in E(Z)\) if and only if \( uv \in E(H) \) and \( v = v' \) or \( u = u' \) and \( uv' \in E(H) \).

Define \( \psi : V(G) \to V(Z) \) by \( \psi(u) = (f(u), g(u)) \) for all \( u \in V(G) \). Note that \( \psi \) is a surjective weak homomorphism from \( G \) to \( Z \). Therefore, \( Z = \psi(G) \).
and since $G$ is connected, also $Z$ is connected by Lemma 2.14. Therefore, $Z \subseteq C \sqcap H$. Since

$$\varepsilon_H(Z) = \min\{d_H(x, y) \mid (x, y) \in V(Z)\}$$

$$= \min\{d_H(f(u), g(u)) \mid u \in V(G)\}$$

$$= m_G(f, g) = r,$$

it follows that $r \in A$. Hence, $C \subseteq A$ and we have proved that $A = C$.

The proof that $D \subseteq B$ is analogous to the proof that $C \subseteq A$ above, with the additional assumption that $f, g$ are surjective opposite weak homomorphisms.

\section{0-span graphs and graphs with equal vertex and edge span variant}

In this section we focus on 0-span graphs; i.e., graphs in which it is impossible to keep a positive safety distance at all points in time in any of the above introduced models. We also construct an infinite family of graphs for which the corresponding vertex and edge variants are equal.

First we give and prove the following characterisations of graphs with strong vertex span, strong edge span, direct vertex span and direct edge span equal to 0.

\begin{theorem}
Let $H$ be any connected graph. The following statements are equivalent.

1. $\sigma^V_E(H) = 0$.
2. $\sigma^E_V(H) = 0$.
3. $|V(H)| = 1$.
\end{theorem}

\begin{proof}
Let $|V(H)| = 1$. Then $V(H) = \{u\}$ for some $u$ and, obviously, $\sigma^V_E(H) = \sigma^E_V(H) = 0$. Next, let $\sigma^E_V(H) = 0$ or $\sigma^E_V(H) = 0$. We show that $|V(H)| = 1$. Suppose that $|V(H)| > 1$. Let $u, v \in V(H)$ such that $uv \in E(H)$.

If $V(H) = \{u, v\}$, then let $Z$ be defined as follows. Let

$$V(Z) = \{(u, v), (v, u)\}$$

and

$$E(Z) = \{(u, v)(v, u)\}.$$ 

Then $\varepsilon_H(Z) = 1$ and, therefore, $\sigma^V_E(H) > 0$ and $\sigma^E_V(H) > 0$.

Let $|V(H)| > 2$, $u_0v_0 \in E(H)$ and let $G$ be the graph defined by $V(G) = \{u_0, v_0\}$ and $E(G) = \{u_0v_0\}$. Also, let $H_1, H_2, \ldots, H_m$ be the components of $H - G$. Since $V(H) \neq \{u_0, v_0\}$, it follows that $m > 0$. For each $i \in \{1, 2, \ldots, m\}$, let

$$U_i = N(u_0) \cap V(H_i)$$

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and

\[ V_i = N(v_0) \cap V(H_i). \]

We define a graph \( Z \) as follows. Let

\[
V(Z) = \left( \bigcup_{i=1}^{m} V(H_i \boxtimes G) \right) \cup \left( \bigcup_{i=1}^{m} V(G \boxtimes H_i) \right) \cup \{(u_0, v_0), (v_0, u_0)\}
\]

and

\[
E(Z) = \left( \bigcup_{i=1}^{m} E(H_i \boxtimes G) \right) \cup \left( \bigcup_{i=1}^{m} E(G \boxtimes H_i) \right) \cup \{(u_0, v_0), (v_0, u_0)\} \cup
\]

\[
\bigcup_{i=1}^{m} \left( \bigcup_{u \in U_i} \{(u, v_0)(u_0, v_0)\} \cup \bigcup_{v \in V_i} \{(v, u_0)(v_0, u_0)\} \right) \cup
\]

\[
\bigcup_{i=1}^{m} \left( \bigcup_{u \in U_i} \{(v_0, u)(u_0, v_0)\} \cup \bigcup_{v \in V_i} \{(u_0, v)(u_0, v_0)\} \right).
\]

Figure 2: Sketch of the construction of the graph \( Z \) in the case of strong spans.

Figure 2 shows an example of how the vertices and edges are added to \( Z \) for any component \( H_i \). It is clear that \( Z \) is connected and that \( p_1(Z) = p_2(Z) = H \) (and therefore also \( p_1(V(Z)) = p_2(V(Z)) = V(H) \)). Since for any vertex \( u \in V(H) \) it holds true that \((u, u) \notin V(Z)\), therefore \( \varepsilon_H(Z) > 0 \). It follows that \( \sigma^V_v(H) > 0 \) and \( \sigma^E_v(H) > 0 \).

**Theorem 4.2.** If \( H \) is the one-vertex graph or \( \text{rad}(H) = 1 \), then

\[ \sigma^E_v(H) = \sigma^V_v(H). \]
Theorem 4.4. Let all connected graphs

Problem 4.3. Hence, we present the following open problem.

Proof. The case when $H$ is the one-vertex graph follows directly from Theorem 4.1. Now let $H$ be such that $\text{rad}(H) = 1$. Also from Theorem 4.1, it follows that $\sigma^{\text{Z}}_{\text{Z}}(H) \neq 0$ and $\sigma^{\text{Z}}_{\text{Z}}(H) \neq 0$. Since $\text{rad}(H) = 1$, using Observation 3.2, we immediately obtain that $\sigma^{\text{Z}}_{\text{Z}}(H) = \sigma^{\text{Z}}_{\text{Z}}(H) = 1$.

Note that for a path $P_n$, for any integer $n$, it holds that $\sigma^{\text{Z}}_{\text{Z}}(P_n) = \sigma^{\text{Z}}_{\text{Z}}(P_n)$. Moreover, for any $n > 1$, $\sigma^{\text{Z}}_{\text{Z}}(P_n) = \sigma^{\text{Z}}_{\text{Z}}(P_n) = 1$. Also, for any $n > 3$, $\text{rad}(P_n) > 1$. Therefore there are graphs $H$ such that $\text{rad}(H) > 1$ and $\sigma^{\text{Z}}_{\text{Z}}(H) = \sigma^{\text{Z}}_{\text{Z}}(H)$. Hence, we present the following open problem.

Problem 4.3. Find all connected graphs $H$ for which $\sigma^{\text{Z}}_{\text{Z}}(H) = \sigma^{\text{Z}}_{\text{Z}}(H)$.

Theorem 4.4. Let $H$ be any connected graph. The following statements are equivalent.

1. $\sigma^{\text{Z}}_{\text{Z}}(H) = 0$.
2. $\sigma^{\text{Z}}_{\text{Z}}(H) = 0$.
3. $|V(H)| = 1$.

Proof. The proof is the same as the proof of Theorem 4.1 with the only difference being in the case $|V(H)| > 2$ in the definition of the graph $Z$. Here the graph $Z$ is defined as follows. Let $u_0v_0 \in E(H)$ and let $G$ be the graph defined by $V(G) = \{u_0, v_0\}$ and $E(G) = \{u_0v_0\}$. Also, let $H_1, H_2, \ldots, H_m$ be the components of $H - G$. Since $V(H) \neq \{u_0, v_0\}$, it follows that $m > 0$. For each $i \in \{1, 2, \ldots, m\}$, let

$$U_i = N(u_0) \cap V(H_i)$$

and

$$V_i = N(v_0) \cap V(H_i).$$

For each $i \in \{1, 2, \ldots, m\}$, each $u \in U_i$ and each $v \in V_i$, let

1. $A_i^u$ be the component of $H_i \times G$ that contains $(u, u_0)$,
2. $A_i^v$ be the component of $H_i \times G$ that contains $(v, v_0)$,
3. $B_i^u$ be the component of $G \times H_i$ that contains $(u_0, u)$ and
4. $B_i^v$ be the component of $G \times H_i$ that contains $(v_0, v)$.

Note, that for all above components the following holds: $p_1(A^u_i) = H_i$, $p_2(A^u_i) = G$, $p_1(B^u_i) = G$ and $p_2(B^u_i) = H_i$.

Let

$$V(Z) = \left( \bigcup_{i=1}^{m} \bigcup_{u \in U_i} (V(A^u_i) \cup V(B^u_i)) \right) \cup \left( \bigcup_{i=1}^{m} \bigcup_{v \in V_i} (V(A^v_i) \cup V(B^v_i)) \right) \cup \{(u_0, v_0), (v_0, u_0)\}.$$
and

\[ E(Z) = \left( \bigcup_{i=1}^{m} \bigcup_{u \in U_i} (E(A^u_i) \cup E(B^u_i)) \right) \cup \left( \bigcup_{i=1}^{m} \bigcup_{v \in V_i} (E(A^v_i) \cup E(B^v_i)) \right) \cup \]

\[ \bigcup_{i=1}^{m} \left( \bigcup_{u \in U_i} \{(u, u_0)(u_0, v_0)\} \right) \cup \left( \bigcup_{v \in V_i} \{(v, v_0)(v_0, u_0)\} \right) \cup \]

\[ \{(u_0, v_0)(v_0, u_0)\}. \]

Figure 3: Sketch of the construction of the graph \( Z \) in the case of direct spans.

Problem 4.6. Find all connected graphs \( H \) for which \( \sigma^x_E(H) = \sigma^x_V(H) \).
Next we give and prove the following characterisation of graphs with the Cartesian vertex span and Cartesian edge span equal to 0.

**Theorem 4.7.** Let $H$ be any connected graph. The following statements are equivalent.

1. $\sigma^\square_E(H) = 0$.
2. $\sigma^\square_V(H) = 0$.
3. There is a positive integer $n$ such that $H$ is an $n$-path.

**Proof.** Let $H = P_n$ be a path on $n$ vertices, for an arbitrary positive integer $n$. Denote the vertices of $P_n$ by $v_0, v_1, \ldots, v_{n-1}$, such that for any $i \in \{0, \ldots, n-2\}$ the vertices $v_i$ and $v_{i+1}$ are adjacent. We show that $\sigma^\square_V(H) = 0$. Using Theorem 3.12 let $Z \subseteq C H \square H$ be such that $p_1(V(Z)) = p_2(V(Z)) = V(H)$ and $\varepsilon_H(Z) = \sigma^\square_H(H)$. Note, the set $\{(v_i, v_j) \mid i < j \text{ and } i, j \in \{0, 1, \ldots, n-1\}\}$ is a cut of $H = P_n \square P_n$ which divides the graph $H = P_n \square P_n$ into two connected components, one with the vertex set $V_0 = \{(v_i, v_j) \mid i < j \text{ and } i, j \in \{0, 1, \ldots, n-1\}\}$ and the other with the vertex set $V_1 = \{(v_i, v_j) \mid i > j \text{ and } i, j \in \{0, 1, \ldots, n-1\}\}$. Towards contradiction suppose $\sigma^\square_V(H) > 0$. This implies that for any $i \in \{0, 1, \ldots, n-1\}$ the vertex $(v_i, v_i)$ does not belong to $V(Z)$. Since $p_1(V(Z)) = V(H)$ there exists a vertex $(v_0, v_j) \in V(Z)$ with $j > 0$. Moreover, $(v_0, v_j)$ belongs to the set $V_\epsilon$. Similarly, since $p_1(V(Z)) = V(H)$ there also exists a vertex $(v_{n-1}, v_l) \in V(Z)$ with $l < n - 1$ and this vertex belongs to $V_\epsilon$. But then $(v_0, v_j)$ and $(v_{n-1}, v_l)$ belong to two distinct connected components, a contradiction to the fact that $Z$ is connected. Therefore, $\sigma^\square_V(H) = 0$ and using Observation 3.11 also $\sigma^\square_E(H) = 0$.

Next, we show that $\sigma^\square_V(H) = 0$ or $\sigma^\square_E(H) = 0$ implies that there exists a positive integer $n$ such that $H$ is an $n$-path. Suppose that $H$ is not an $n$-path for any positive integer $n$. This means that $H$ contains a cycle or it is a tree that contains a vertex of degree at least 3.

Suppose $H$ contains a cycle. Let $C$ be a smallest cycle in $H$, say that it is induced by the vertices $V(C) = \{c_0, c_1, \ldots, c_k\}$, for some $k > 2$. Moreover, let the notation be chosen such that $c_i c_{i+1} \in E(C)$ for all $i \in \{0, 1, \ldots, k\}$ where the indices are computed modulo $k + 1$. Also, let $H_1, H_2, \ldots, H_m$ be the components of $H - C$. Clearly, $m \geq 0$. For each $i \in \{1, \ldots, m\}$ and for each $j \in \{0, 1, \ldots, k\}$ let $U_{i,j} = V(H_i) \cap N(c_j)$.

Note that $U_{i,j}$ may be an empty set. We define a graph $Z$ as follows, see Figure 4. Let

$$V(Z) = \left( \bigcup_{i=1}^{m} V(H_i \square C) \right) \cup \left( \bigcup_{i=1}^{m} V(C \square H_i) \right) \cup \left( \bigcup_{i=1}^{k} \{(c_0, c_1), (c_i, c_0)\} \right) \cup \{(c_k, c_1), (c_1, c_k)\}$$

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and

\[ E(Z) = \left( \bigcup_{i=1}^{m} E(H_i \square C) \right) \cup \left( \bigcup_{i=1}^{m} E(C \square H_i) \right) \cup \\
\left( \bigcup_{i=1}^{k-1} \{ (c_0, c_i)(c_0, c_{i+1}), (c_i, c_0)(c_{i+1}, c_0) \} \right) \cup \\
\{ (c_k, c_0)(c_k, c_1), (c_0, c_k)(c_1, c_k), (c_1, c_0)(c_1, c_k), (c_0, c_1)(c_k, c_1) \} \cup \\
\bigcup_{i=0}^{m} \left( \bigcup_{j=1}^{k} \left( \bigcup_{v \in U_{i,j}} \{ (v, c_0)(c_{j}, c_0) \} \right) \right) \cup \\
\bigcup_{i=0}^{m} \left( \bigcup_{v \in U_{i,0}} \{ (v, c_1)(c_0, c_1) \} \right) \cup \\
\bigcup_{i=0}^{m} \left( \bigcup_{v \in U_{i,j}} \{ (c_0, v)(c_0, c_j) \} \right) \cup \\
\bigcup_{i=0}^{m} \left( \bigcup_{v \in U_{i,0}} \{ (c_1, v)(c_1, c_0) \} \right).
\]

By the construction \( Z \) is a connected subgraph of \( H \square H \), moreover \( p_1(Z) = p_2(Z) = H \) (and therefore also \( p_1(V(Z)) = p_2(V(Z)) = V(H) \)). Since for any \( u \in V(H) \) the vertex \( (u, u) \notin V(Z) \) it follows that \( \varepsilon_H(Z) > 0 \). And therefore \( \sigma^0_V(H) > 0 \) and \( \sigma^0_E(H) > 0 \).

Finally, let \( H \) be a tree that contains a vertex of degree at least three, say \( u_0 \), and let \( u_1, u_2, u_3 \) be three distinct neighbours of \( u_0 \). Note, since \( H \) is a tree, the vertices \( u_1, u_2 \) and \( u_3 \) induce a graph with no edges. Let \( G = \langle \{u_0, u_1, u_2, u_3\} \rangle_H \) and let \( H_1, H_2, \ldots, H_m \) be the components of \( H - G \), for some \( m \geq 0 \). For each \( i \in \{1, \ldots, m\} \) and for each \( j \in \{0, 1, 2, 3\} \) let

\[ U_{i,j} = V(H_i) \cap N(u_j). \]

Again, we construct a connected subgraph \( Z \) of \( H \square H \) with \( \varepsilon_H(Z) > 0 \) as follows (see Figure [5]). Let

\[ V(Z) = \left( \bigcup_{i=1}^{m} V(H_i \square G) \right) \cup \left( \bigcup_{i=1}^{m} V(G \square H_i) \right) \cup \\
(V(G) \times V(G)) \setminus \left( \bigcup_{u \in V(G)} \{ (u, u) \} \right).
\]
and

\[
E(Z) = \left( \bigcup_{i=1}^{m} E(H_i \Box G) \right) \cup \left( \bigcup_{i=1}^{m} E(G \Box H_i) \right) \cup \\
E \left( (G \Box G) \setminus \bigcup_{u \in V(G)} \{(u, u)\} \right) \cup \\
\bigcup_{i=0}^{m} \left( \bigcup_{j=0}^{2} \bigcup_{v \in U_{i,j}} \{(v, u_3)_{(u_j, u_3)}\} \right) \cup \bigcup_{i=0}^{m} \left( \bigcup_{v \in U_{i,3}} \{(v, u_0)_{(u_3, u_0)}\} \right) \cup \\
\bigcup_{i=0}^{m} \left( \bigcup_{j=0}^{2} \bigcup_{v \in U_{i,j}} \{(u_3, v)_{(u_3, u_j)}\} \right) \cup \bigcup_{i=0}^{m} \left( \bigcup_{v \in U_{i,3}} \{(u_0, v)_{(u_0, u_3)}\} \right)
\]

Again, by the construction, Z is a connected subgraph of \(H \Box H\), moreover 
\(p_1(Z) = p_2(Z) = H\) (and therefore also \(p_1(V(Z)) = p_2(V(Z)) = V(H)\)). Since
for any $u \in V(H)$ the vertex $(u, u) \not\in V(Z)$ it follows that $\varepsilon_H(Z) > 0$. And therefore $\sigma^\cap V(H) > 0$ and $\sigma^\cap E(H) > 0$. 

**Theorem 4.8.** Let $H$ be a connected graph. If $H$ is a path or $\text{rad}(H) = 1$, then

$$\sigma^\cap E(H) = \sigma^\cap V(H).$$

**Proof.** The case where $H$ is a path follows directly from Theorem 4.7. Assume that $H$ is not a path and $\text{rad}(H) = 1$. From Theorem 4.7 it follows that $\sigma^\cap V(H) \neq 0$ and $\sigma^\cap E(H) \neq 0$. Since $\text{rad}(H) = 1$, using Observation 3.11 we immediately obtain that $\sigma^\cap V(H) = \sigma^\cap E(H) = 1$.

**Problem 4.9.** Find all connected graphs $H$ for which $\sigma^\cap E(H) = \sigma^\cap V(H)$.

## 5 Open problems

We conclude the paper with the following open problems.
Problem 5.1. Let $n$ be a positive integer. Characterise all connected graphs $H$ with the strong vertex span $\sigma^v_V(H) = n$.

Problem 5.2. Let $n$ be a positive integer. Characterise all connected graphs $H$ with the direct span $\sigma^d_E(H) = n$.

Problem 5.3. Let $n$ be a positive integer. Characterise all connected graphs $H$ with the direct vertex span $\sigma^d_V(H) = n$.

Problem 5.4. Let $n$ be a positive integer. Characterise all connected graphs $H$ with the direct edge span $\sigma^d_E(H) = n$.

Problem 5.5. Let $n$ be a positive integer. Characterise all connected graphs $H$ with the Cartesian vertex span $\sigma^\Box_V(H) = n$.

Problem 5.6. Let $n$ be a positive integer. Characterise all connected graphs $H$ with the Cartesian edge span $\sigma^\Box_E(H) = n$.

Problem 5.7. Let $H$ be any connected graph with the strong edge span $\sigma^E_E(H)$. Find an efficient algorithm to find $Z \subseteq C \times H \times H$ such that $p_1(Z) = p_2(Z) = H$ and $\varepsilon_H(Z) = \sigma^E_E(H)$.

Problem 5.8. Let $H$ be any connected graph with the strong vertex span $\sigma^V_V(H)$. Find an efficient algorithm to find $Z \subseteq C \times H \times H$ such that $p_1(V(Z)) = p_2(V(Z)) = V(H)$ and $\varepsilon_H(Z) = \sigma^V_V(H)$.

Problem 5.9. Let $H$ be any connected graph with the direct edge span $\sigma^E_E(H)$. Find an efficient algorithm to find $Z \subseteq C \times H \times H$ such that $p_1(Z) = p_2(Z) = H$ and $\varepsilon_H(Z) = \sigma^E_E(H)$.

Problem 5.10. Let $H$ be any connected graph with the direct vertex span $\sigma^V_V(H)$. Find an efficient algorithm to find $Z \subseteq C \times H \times H$ such that $p_1(V(Z)) = p_2(V(Z)) = V(H)$ and $\varepsilon_H(Z) = \sigma^V_V(H)$.

Problem 5.11. Let $H$ be any connected graph with the Cartesian vertex span $\sigma^\Box_V(H)$. Find an efficient algorithm to find $Z \subseteq C \times H \times H$ such that $p_1(Z) = p_2(Z) = H$ and $\varepsilon_H(Z) = \sigma^\Box_V(H)$.

Problem 5.12. Let $H$ be any connected graph with the Cartesian edge span $\sigma^\Box_E(H)$. Find an efficient algorithm to find $Z \subseteq C \times H \times H$ such that $p_1(V(Z)) = p_2(V(Z)) = V(H)$ and $\varepsilon_H(Z) = \sigma^\Box_E(H)$.

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References
